Optimal control for holonomic and Pfaffian nonholonomic systems

John E. Hurtado  
Texas A & M Univ., College Station

John L. Junkins  
Texas A & M Univ., College Station


Holonomic and Pfaffian nonholonomic systems are dynamical systems whose generalized coordinates and velocities are subject to smooth constraints. These systems are usually described by second order differential equations of motion and algebraic equations of constraint. We present an optimal control formulation for these systems that utilize the multiplier rule to append both the equations of motion and equations of constraint directly to the performance index. Variational calculus techniques are used to obtain the necessary conditions, and we find that, like the original state dynamical system, the costate system also represents a differential-algebraic constrained dynamical system. To numerically solve the evolution of the state and costate coupled set of differential-algebraic equations, we propose a special form of an augmented Lagrangian penalty method. Several examples are included to evaluate the merits of the new methods introduced. (Author)
Optimal Control for Holonomic and Pfaffian Nonholonomic Systems

John E. Hurtado* and John L. Junkins†
Department of Aerospace Engineering
Texas A&M University, College Station, TX 77843

Abstract
Holonomic and Pfaffian nonholonomic systems are dynamical systems whose generalized coordinates and velocities are subject to smooth constraints. These systems are usually described by second order differential equations of motion and algebraic equations of constraint. We present an optimal control formulation for these systems that utilizes the multiplier rule to append both the equations of motion and equations of constraint directly to the performance index. Variational calculus techniques are used to obtain the necessary conditions, and we find that, like the original state dynamical system, the costate system also represents a differential-algebraic constrained dynamical system. To numerically solve the evolution of the state and costate coupled set of differential-algebraic equations, we propose a special form of an augmented Lagrangian penalty method. Several examples are included to evaluate the merits of the new methods introduced.

Introduction
The differential equations that best model a dynamical system are often subject to constraints. A common example is that the generalized coordinates and velocities of the system are related. How to optimally control these constrained dynamical systems is a question of both academic and practical interest.

The question is academically appealing, because the optimal control problem is essentially a decision problem that asks what are the 'best' choices that one could make to steer the motion of a system through an infinity of possible state transitions. Determining the best controls as the solution to the decision problem is an approach that ideally "... has the virtues of simplicity, preciseness, elegance, and ... mathematical tractability." Admittedly, tractability and simplicity are often lofty and sometimes unattainable goals.

Refined methodologies for solving the optimal control problem is clearly of practical interest because the control of generally nonlinear, multi-input, multi-output systems is typically not amenable to solutions by classical methods. The formulations and algorithms presented herein provide one solution approach, and furthermore, these optimal control algorithms can be used to design reference maneuvers to be tracked through feedback control.

In this work, we consider optimal control formulations for systems that are subject to holonomic and nonholonomic constraints. Immediately below we further introduce the subject matter and discuss some notable previous studies. In the sections that follow, we discuss the optimal control formulation and our solution method for these systems, and include a set of illuminating example problems.

Holonomic systems
A significant class of problems in analytical dynamics are formulated such that they are represented as systems subject to holonomic constraints. That is, the generalized coordinates describing the system are not independent, but rather are related through nonlinear algebraic equations. These systems are commonly referred to as holonomic systems, and examples include multi-body robotic and satellite systems wherein the joint angles between substructures may undergo large rotations.

In theory, the excess coordinates of a holonomic system may be eliminated, resulting in a 'constraint free' dynamical system. In fact, traditional optimal control formulations for holonomic systems begin with transforming the governing equations into a constraint free form by one of three methods, before the usual procedures for arriving at the necessary conditions for optimal control are applied.

In the first method, the holonomic constraints are used to algebraically eliminate redundant coordinates and the equations of motion are formulated using a globally valid minimal coordinate description of the system. For most complex structures, however, minimal coordinate descriptions are difficult to generate. In a second method, locally equivalent to the first, the generalized coordinates undergo a judicious nonlinear coordinate transformation. In these new coordinates, the constraints are trivially satisfied leaving a subset of differential equations that are not subject to constraint forces. We stress, however, that the nonlinear
coordinate transformations of this method are only locally valid. And in a third approach, one differentiates the holonomic constraint equations and uses the equations of motion to numerically or algebraically eliminate the Lagrange multipliers in favor of nonlinear functions of the generalized coordinates, velocities and controls. This approach is known as either a range space or null space projection formulation depending upon the particular method of elimination used.\(^9\)

As mentioned, subsequent to these manipulations, the usual procedures for deriving expressions for the optimal control may be applied. In this study, a different avenue towards the optimal control of these systems is pursued. The approach is driven by the desire to avoid the algebraic effort required to produce 'constraint free' descriptions of these systems. In our approach the equations that describe the motion of the system together with the equations that define the constraints are appended to a performance index using the multiplier rule. Variational calculus techniques applied to this constrained system then yield an adjoint, or costate, system that, like the original dynamical system, is also constrained. A novel form for the state and costate system results which constitute a (one-way) coupled set of differential-algebraic equations, and we propose an augmented Lagrangian penalty method\(^{10}\) to numerically solve them. This penalty solution approach has recently shown considerable promise in solving the differential-algebraic equations that arise in multi-body dynamic formulations.\(^{11,12}\)

**Pfaffian nonholonomic systems**

We frequently encounter nonintegrable equations of constraint that involve nonlinear functions of the generalized coordinates acting on a linear combination of the generalized velocities. Systems subject to these types of constraint are referred to as Pfaffian nonholonomic systems,\(^{13}\) and one classic example is a wheel rolling without slip. The important difference between holonomic and Pfaffian nonholonomic systems is that for the latter, one no longer has the option of transforming the equations to a minimal coordinate description: we are forced to work with a number of generalized coordinates exceeding the number of degrees of freedom. We note that more general nonholonomic systems, involving inequality constraints or constraints described by nonlinear functions of the generalized velocities, are not considered in this work.

Control of Pfaffian nonholonomic systems has received some attention in the recent past. Bloch, Reyhanoglu, and McClamroch have largely focused on the issues of controllability and stabilizability,\(^{14,15,16}\) and on open loop control methods using geometric phases.\(^{17}\) Their methods rely upon locally valid nonlinear coordinate transformations that decompose the system of equations into a subset that are independent of the constraint forces.

Traditional methods for optimal control of Pfaffian nonholonomic systems may begin after invoking these nonlinear coordinate transformations, or after using one of the projection methods that was discussed for holonomic systems. Our approach for optimal control of nonholonomic systems closely follows that taken for holonomic systems, but requires a modification of the augmented Lagrangian method.

**Governing equations**

As mentioned, holonomic systems are dynamical systems that are formulated such that the generalized coordinates are not independent, but rather are related through nonlinear algebraic equations of the form \(\varphi_0(q) = 0\). Because the coordinates are not independent, when formulating the governing differential equations of motion, one must account for the constraint forces that restrict the time and space evolution of the system. This is done by representing the constraint forces as \(\frac{\partial \varphi_0}{\partial q_k} \lambda_0\), where \(\lambda_0\) are elements of a time varying vector of Lagrange multipliers that enforce the holonomic constraints of the system. Thus, with appropriate definitions, the governing differential-algebraic equations of motion for a holonomic system take the form\(^{18}\)

\[
\ddot{q}_i + h_i(q, \dot{q}) + g_i(q) = b_{im}(q) u_m + d_{io}(q) \lambda_0 \quad (1)
\]

subject to \(\varphi_0(q) = 0\) \(\quad (2)\)

In Eqs. (1), \(h_i(q, \dot{q})\) and \(g_i(q)\) are generally nonlinear functions of the generalized coordinates and velocities, \(b_{im}(q)\) is the control influence matrix, \(u_m\) represents the controls, and \(d_{io}(q) \lambda_0\) represent the constraint forces. Pfaffian constraints involve nonlinear functions of the generalized coordinates acting on a linear combination of the generalized velocities, and may be expressed by \(\phi_{oi}(q) \ddot{q}_i = 0\). If these relationships are integrable, then the constraints are holonomic and can be reduced to the form stated above; if they are nonintegrable, the constraints are nonholonomic, and the system requires more coordinates for its description than there are degrees of freedom.\(^7\) These systems are referred to as nonholonomic systems. The constraint forces that restrict the time and space evolution of a nonholonomic system must be accounted for when developing the equations of motion. Thus, the governing differential-algebraic equations of motion for a nonholonomic system take the form\(^{18}\)

\[
\ddot{q}_i + h_i(q, \dot{q}) + g_i(q) = b_{im}(q) u_m + d_{io}(q) \lambda_0 \quad (3)
\]

subject to \(\phi_{oi}(q) \ddot{q}_i = 0\) \(\quad (4)\)

We emphasize that for holonomic and Pfaffian nonholonomic dynamical systems the governing set of differential-algebraic equations must be solved simultaneously for the unknown vectors \(q(t), \dot{q}(t),\) and \(\lambda(t)\).
Optimal control formulation

The necessary conditions for optimal control are almost universally derived with the equations of motion in first order form. In the formulations below, we retain the second order form of the differential equations of motion, and use variational calculus techniques to arrive at the costate system of equations.

For holonomic and Pfaffian nonholonomic systems, we consider the familiar quadratic performance measure, modified to include a penalty on control derivatives, given by

$$J = \int_{t_0}^{t_f} \left[ \frac{1}{2} Q_{ij} \dot{q}_i \dot{q}_j + \frac{1}{2} Q^{ij} \dot{q}_i \dot{q}_j + \frac{1}{2} R_{im} u_i u_m + \frac{1}{2} P_{im} \dot{u}_i u_m \right] dt$$  \hspace{1cm} (5)

where $Q^{ij}$, $Q_{ij}$, $R_{im}$, and $P_{im}$ are elements of weight matrices with the usual properties.\(^{18}\) We have included the control derivatives in the performance index so that we can specify the controls at the initial and final times: this is very beneficial in robotic and satellite problems to avoid initial and final jump discontinuities of the controls and to smooth the overall control profiles.\(^{19}\)

**Holonomic systems**

We seek to minimize Eq. (5) subject to the constraints given by Eqs. (1) and (2). Using the multiplier rule, we append the dynamical system and constraint equations to the performance index, and denote $v_i$ and $\gamma_e$ as elements of the time varying vectors of Lagrange multipliers for the differential state equations and the algebraic constraint equations, respectively.

Limiting ourselves to smooth, unbounded controls, the necessary conditions for finding the optimal trajectory and resulting controls follow from requiring that the first variation of the augmented performance index equal zero. Invoking the usual arguments regarding the smoothness of the state dynamical system, the arbitrariness of the Lagrange multipliers $v_i$ and $\gamma_e$, and the independence and arbitrariness of the variations of the controls,\(^{18}\) leads us to the necessary conditions for optimal control of holonomic systems.\(^{18}\)

State dynamical system:

$$\ddot{q}_i + h_i(q, \dot{q}) + g_i(q) = b_{im}(q) u_m + d_{ie}(q) \lambda_e$$ \hspace{1cm} (6a)

subject to $\varphi_e(q) = 0$ \hspace{1cm} (6b)

Adjoint dynamical system:

$$\ddot{\gamma}_i = -\frac{d}{dt} (v_i \frac{\partial h_i}{\partial \gamma_j}) + v_i \left( \frac{\partial h_i}{\partial \dot{q}_j} + \frac{\partial g_i}{\partial \dot{q}_j} - \frac{\partial b_{im}}{\partial \dot{q}_j} u_m \right) - \frac{\partial d_{ie}}{\partial \gamma_j} \lambda_e$$ \hspace{1cm} (7a)

subject to $v_i d_{ie}(q) = 0$ \hspace{1cm} (7b)

Optimality conditions:

$$P_{im} \ddot{u}_i - R_{im} u_i = v_i b_{im}$$  \hspace{1cm} (8)

The boundary terms must also vanish independently; this gives the transversality conditions

$$[Q^{ij} \dot{q}_i + \dot{v}_i - \frac{\partial h_i}{\partial \dot{q}_j} \delta q_j]_{t_0} = 0, \quad v_i \frac{\partial q_i}{\partial \gamma_j} |_{t_0} = 0$$

$$\text{and } P_{im} \ddot{u}_i \delta u_m |_{t_0} = 0$$  \hspace{1cm} (9a-c)

Of course, like the differential equations, these transversality conditions must be satisfied subject to the constraint relationships. We emphasize that Eqs. (6) through (9) form a two-point boundary-value problem, described by a coupled set of differential-algebraic equations.

**Pfaffian nonholonomic systems**

The necessary conditions for optimal control of Pfaffian nonholonomic systems are derivable using the same techniques that are used for holonomic systems. The necessary conditions are given below and are similar to those listed for holonomic systems. Note, however, that the form of the constraint equation for Pfaffian nonholonomic systems (see Eq. (4)) has slightly altered the right hand side of the adjoint differential equations. The necessary conditions for optimal trajectories follow\(^{18}\)

State dynamical system:

$$\ddot{q}_i + h_i(q, \dot{q}) + g_i(q) = b_{im}(q) u_m + d_{ie}(q) \lambda_e$$ \hspace{1cm} (10a)

subject to $\varphi_e(q) = 0$ \hspace{1cm} (10b)

Adjoint dynamical system:

$$\ddot{\gamma}_i = -\frac{d}{dt} (v_i \frac{\partial h_i}{\partial \gamma_j}) + v_i \left( \frac{\partial h_i}{\partial \dot{q}_j} + \frac{\partial g_i}{\partial \dot{q}_j} - \frac{\partial b_{im}}{\partial \dot{q}_j} u_m \right)$$

$$- \frac{\partial d_{ie}}{\partial \gamma_j} \lambda_e = Q^{ij} q_i - Q_{ij} \dot{q}_i$$

$$+ \frac{\partial \varphi_e}{\partial \dot{q}_j} \dot{q}_i - \frac{d}{dt} (\gamma_e \varphi_{ij})$$ \hspace{1cm} (11a)

subject to $v_i d_{ie}(q) = 0$ \hspace{1cm} (11b)

The differential optimality conditions and transversality conditions have the same form as those given for holonomic systems.\(^{18}\)

In this section we have found that the necessary conditions for optimal control of holonomic and Pfaffian nonholonomic systems form a two-point boundary-value problem (TPBVP), described by a coupled set of differential-algebraic equations (DAEs). In the next section we extend recent algorithms for initial value problems involving DAEs to establish a numerical solution strategy to solve this two-point boundary-value problem involving DAEs.
Augmented Lagrangian method in optimal control

In most problems of interest, the set of equations defining the necessary conditions for optimal control must be solved numerically. For the constrained dynamical systems that we consider in this current work, this means not only a numerical solution for solving the TPBVP, but perhaps more challenging, a numerical solution strategy for solving the time and space evolution of the coupled set of DAEs. In fact, the unique form of the DAEs and the TPBVP pose a novel challenge, and we have established a very effective solution strategy.

While the main purpose of the present section is to identify a method for solving the coupled DAEs, we pause to mention the numerical solution strategy for solving the TPBVP. A number of numerical methods can be used to solve TPBVPs arising in optimal control, and for the examples below, we elect to use a shooting method. A thorough discussion of the classical shooting method may be found in reference 19.

Recall, the aim of our optimal control formulation for holonomic systems was to avoid the algebraic effort required to obtain constraint free descriptions of these systems. This aim gave rise to the coupled DAEs that make up the necessary conditions. Similar statements hold true regarding the optimal control formulation for Pfaffian nonholonomic systems. Numerical solution strategies for initial value problems involving DAEs have been a focus of research for some years, and recently a generalized penalty solution method has shown considerable promise.

While historically used to obtain solutions to time independent problems, augmented Lagrangian methods have recently been extended to address the initial value DAEs that arise in multi-body dynamic formulations. Moreover, analysis for very general nonlinear dynamical systems has been conducted which establishes not only conditions that guarantee convergence, but also bounds on the rate of convergence of the method.

Augmented Lagrangian methods are iterative and involve approximating the constraint functions and the Lagrange multipliers that enforce them. The approximate multipliers are updated based upon a measure of constraint violation. When applied to constrained dynamical systems, the solution process can be viewed as quasi-static. Specifically, an iteration process is triggered at each time step wherein, for obtaining a correction equation, the positions and velocities are held constant while the accelerations are considered a static quantity. Upon completion, the iteration process yields the accelerations and the Lagrange multipliers composing the constraint forces, hence integration to the next time step can proceed. The iteration process is outlined below and more fully explained in reference 18. We first focus on holonomic systems, followed by the application to Pfaffian nonholonomic systems.

Holonomic systems

The key for using augmented Lagrangian methods in optimal control of holonomic systems lies in locking at the state dynamical system first, and then turning to the adjoint system. Because these equations lend themselves to this staggered solution approach, we say the state and adjoint DAEs are 'one-way' coupled.

State dynamical system

The equations that govern the state dynamical system may be written (cf. Eqs. (6))

\[
\ddot{q}_i = X_i(q, \dot{q}, u) + d_{io}(q) \lambda_0
\]

subject to \( \varphi_\circ(q) = 0 \) (12a)

where \( X_i \) contains nonlinear terms involving the generalized coordinates and velocities, and the control input terms. The iterative scheme triggered at each time step for the state dynamical system is based upon the following approximation to the true system:

\[
\ddot{q}_i^n = X_i(q, \dot{q}, u) + d_{io} \lambda_0^n - \frac{1}{\varepsilon} d_{io} \Phi_\circ^n
\]

(12b)

\[
\lambda_0^{n+1} = \lambda_0^n - \frac{1}{\varepsilon} \Phi_\circ^n \quad \text{with} \quad \lambda_0^n = 0
\]

(13a)

where

\[
\Phi_\circ^n \equiv \left[ \frac{\partial \varphi_\circ}{\partial \dot{q}_i} \dot{q}_i + \frac{d}{dt} \left( \frac{\partial \varphi_\circ}{\partial \dot{q}_i} \right) \dot{q}_i + 2\zeta \omega \varphi_\circ + \omega^2 \varphi_\circ \right]
\]

(13b)

\( \ddot{q}_i^n \) and \( \lambda_0^n \) represent current approximations to the true accelerations and Lagrange multipliers, while \( \Phi_\circ^n \), which comprises the holonomic constraints and their derivatives, represents a measure of constraint violation. Furthermore, \( n \) is the iteration number, \( \varepsilon > 0 \) is a penalty factor, and constants \( \zeta, \omega \geq 0 \) represent a damping factor and frequency associated with the constraint penalty.

It is important to note that the terms in \( X_i \), and all but the leading terms in the definition of \( \Phi_\circ^n \), are composed entirely of quantities that are held constant over the iteration process. To establish the convergence of this iterative process to the true accelerations and Lagrange multipliers, instead of Eqs. (12), we express the true dynamical system by

\[
\ddot{q}_i = X_i(q, \dot{q}, u) + d_{io} \lambda_0 - \frac{1}{\varepsilon} d_{io} \Phi_\circ
\]

(14a)

(14b)

where

\[
\Phi_\circ \equiv \left[ \frac{\partial \varphi_\circ}{\partial \dot{q}_i} \dot{q}_i + \frac{d}{dt} \left( \frac{\partial \varphi_\circ}{\partial \dot{q}_i} \right) \dot{q}_i + 2\zeta \omega \varphi_\circ + \omega^2 \varphi_\circ \right]
\]
This form is equivalent to Eqs. (12) because, for the true system, the constraints are exactly satisfied. Subtracting Eqs. (14) from (13) at some arbitrary instant of time $t$ yields

$$[\delta ij + \frac{1}{\epsilon} d_{ij} \frac{\partial \varphi_o}{\partial \eta} (\tilde{q}^n - \tilde{q}_j) = d_{ij} (\lambda^n_o - \lambda_o) \quad (15)$$

$$\lambda^{n+1}_o - \lambda_o = \lambda^n_o - \lambda_o - \frac{1}{\epsilon} \frac{\partial \varphi_o}{\partial \eta_j} (\tilde{q}^n_j - \tilde{q}_j) \quad (16)$$

Motivated by Theorems 2.2 and 2.3 of Glowinski and Le Tellac\textsuperscript{10} (pp. 49-57), convergence of the iteration scheme may be established:\textsuperscript{18}

Claim: $\lim_{n \to +\infty} \tilde{q}^n_j = \tilde{q}_j$.

Proof: (sketch) Performing the inner product of Eqs. (16) with itself and using Eqs. (15) reveals that $\lim_{n \to +\infty} (\lambda^n_o - \lambda_o)^2 = 0$. From this it follows that $\lim_{n \to +\infty} \tilde{q}^n_j = \tilde{q}_j$.

Claim: $\lim_{n \to +\infty} \lambda^n_o = \lambda_o$.

Proof: (sketch) This proof immediately follows from the first claim, and from the assumption that singular configurations of the system are not encountered. □

Thus, at time $t$, this iterative scheme provides the accelerations and the Lagrange multipliers composing the constraint forces; furthermore, by letting $t$ be the initial time $t_0$, we have established convergence for the entire evolution of the system. These remarks, of course, ignore the effects of finite dimensional arithmetic which corrupt, in a problem dependent way, the actual performance of this approach.

**Adjoint dynamical system**

Having established theoretical convergence to the true accelerations and Lagrange multipliers of the state dynamical system at time $t$, we now focus on the equations defining the adjoint system. This discussion strongly mimics the previous one. The equations that govern the adjoint system may be written (cf. Eqs. (7))

$$\ddot{v}_i = Y_i (q, \dot{q}, \ddot{q}, u, \lambda, v, \dot{v}) + \frac{\partial \varphi_o}{\partial q_i} \gamma_o \quad (17a)$$

subject to $v_j d_{j0}(q) = 0 \quad (17b)$

where $Y_i$ contains nonlinear terms involving $q, \dot{q}$, their derivatives, $u$ and $\lambda$. An approximation to the true adjoint system is given by

$$\ddot{v}_i^n = Y_i (q, \dot{q}, \ddot{q}, u, \lambda, v, \dot{v}) + \frac{\partial \varphi_o}{\partial q_i} \gamma_o^n \quad (18a)$$

$$\gamma_o^{n+1} = \gamma_o^n - \frac{1}{\epsilon} \psi_o^n \quad \text{with} \quad \gamma_o^0 = 0 \quad (18b)$$

where

$$\psi_o(q, v) \overset{\text{def}}{=} v_j d_{j0}(q)$$

$$\psi_o^n \overset{\text{def}}{=} \tilde{v}_j^n d_{j0} + 2 \tilde{v}_j \frac{\partial d_{j0}}{\partial q_i} \dot{q}_i + v_j \left[ \frac{\partial^2 d_{j0}}{\partial q_i \partial q_k} \ddot{q}_k \ddot{q}_i \right.$$

$$\left. + \frac{\partial d_{j0}}{\partial q_i} \dot{q}_i \right] + 2\omega \psi_o + \omega^2 \psi_o$$

$\tilde{v}_i^n$ and $\gamma_o^n$ represent current approximations to the true accelerations and Lagrange multipliers of the adjoint system, while $\psi_o^n$ represents a measure of constraint violation. The other parameters are defined as they were in the previous discussion.

It is important to note that the terms forming $Y_i$, and all but the leading terms in the definition of $\psi_o^n$, are composed entirely of quantities that are either held constant over the iteration process, or have already been converged upon from the augmented Lagrangian method performed on the state system: herein lies the benefit of the one-way coupling which has allowed us to operate on the state system first, completely independent of the adjoint system. To establish the convergence of this iterative process to the true adjoint accelerations and Lagrange multipliers, instead of Eqs. (17), we express the true adjoint system by

$$\ddot{v}_i = Y_i (q, \dot{q}, \ddot{q}, u, \lambda, v, \dot{v}) + \frac{\partial \varphi_o}{\partial q_i} \gamma_o$$

$$\gamma_o = \gamma_o - \frac{1}{\epsilon} \psi_o \quad (19a)$$

$$\gamma_o = \gamma_o - \frac{1}{\epsilon} \psi_o \quad (19b)$$

where

$$\psi_o \overset{\text{def}}{=} \tilde{v}_j d_{j0} + 2 \tilde{v}_j \frac{\partial d_{j0}}{\partial q_i} \dot{q}_i + v_j \left[ \frac{\partial^2 d_{j0}}{\partial q_i \partial q_k} \ddot{q}_k \ddot{q}_i \right.$$

$$\left. + \frac{\partial d_{j0}}{\partial q_i} \dot{q}_i \right] + 2\omega \psi_o + \omega^2 \psi_o$$

Subtracting Eqs. (19) from (18) yields

$$[\delta ij + \frac{1}{\epsilon} d_{ij} \frac{\partial \varphi_o}{\partial \eta} (\tilde{v}^n_j - \tilde{v}_j) = \frac{\partial \varphi_o}{\partial \eta_j} (\gamma^n_o - \gamma_o)$$

$$\gamma^{n+1}_o - \gamma_o = \gamma^n_o - \gamma_o - \frac{1}{\epsilon} d_{ij} (\tilde{v}^n_j - \tilde{v}_j) \quad (19c)$$

From this point, convergence of $\tilde{v}^n_j \to \tilde{v}_j$ and $\gamma^n_o \to \gamma_o$ at time $t$ may be demonstrated by following the arguments presented for the state dynamical system.\textsuperscript{18}

**Implementation**

Thus the iterative process of the augmented Lagrangian method provides $\tilde{q}, \tilde{v}, \lambda$ and $\gamma$ at a particular time, and integration to the next time step can proceed. Implementing the routine for the state dynamical system at time $t$ reads:

1. With $\lambda^0 = 0$, solve Eq. (13a) for $\tilde{q}^0$.  

American Institute of Aeronautics and Astronautics
2. Update $\lambda$ using Eq. (13b).
3. Substitute updated $\lambda$ back into Eq. (13a).
4. Continue until $||\Phi^a||$ is less than some prescribed tolerance.

This routine applies, mutatis mutandis, to the adjoint system.

One final comment regarding the augmented Lagrangian method pertains to the rate of convergence of the method. Menon$^{12}$ shows that for multi-body dynamical systems, the parameter $\epsilon$ governs the rate of convergence. Specifically, at each fixed-time step, the rate of convergence of the constraint violation penalty, $\Phi$, from one iteration to the next, is of the order $\epsilon$.

While we have not rigorously addressed the rate of convergence of the augmented Lagrangian method in optimal control, it is anticipated that the same statements apply to the state and adjoint dynamical systems. That is, the rate of convergence of the state system constraint violation penalty, $\Phi$, and the costate system constraint violation penalty, $\Psi$, is of the order $\epsilon$.

Pfaffian nonholonomic systems

The application of augmented Lagrangian penalty methods in optimal control of holonomic systems was outlined in the last section. There we found that it was critical to exploit the one-way coupling between the state and adjoint systems. The application of the method to Pfaffian nonholonomic systems is essentially the same, and is highlighted below. The main difference is that the adjoint system must be partitioned to reveal yet another one-way coupling: this time within the adjoint system.

**State dynamical system**

Beginning with the state dynamical system, we write the governing equations as (cf. Eqs. (10))

$$\dot{q}_i = X_i (q, \dot{q}, u) + d_i \lambda_0$$

subject to $\varphi_0 (q, \dot{q}) \stackrel{def}{=} \phi_0 (q) \dot{q}_i = 0$

where $X_i$ contains nonlinear terms involving the generalized coordinates and velocities, and the control input terms. An approximation to the true system is given by

$$\ddot{q}_i^n = X_i (q, \dot{q}_i, u) + d_i \lambda_0 - \frac{1}{\epsilon} d_i \Phi_0$$

$$\lambda_0^{n+1} = \lambda_0^n - \frac{1}{\epsilon} \Phi_0^n \quad \text{with} \quad \lambda_0^n = 0$$

where

$$\Phi_0^n \stackrel{def}{=} \varphi_0^n + \omega \varphi_0 = [\phi_{ej} \dot{q}_j^n + \frac{d \phi_{ej}}{dt} \dot{q}_j + \omega \phi_{ej} \dot{q}_j]$$

To establish the convergence of the iterative process to the true accelerations and Lagrange multipliers of the state system, instead of Eqs. (20), we consider an equivalent form given by

$$\ddot{q}_i = X_i (q, \dot{q}_i, u) + d_i \lambda_0 - \frac{1}{\epsilon} d_i \Phi_0$$

$$\lambda_0 = \lambda_0 - \frac{1}{\epsilon} \Phi_0$$

where

$$\Phi_0 \stackrel{def}{=} \varphi_0 + \omega \varphi_0 = [\phi_{ej} \dot{q}_j + \frac{d \phi_{ej}}{dt} \dot{q}_j + \omega \phi_{ej} \dot{q}_j]$$

From this point, convergence of $\dot{q}_i^n \rightarrow \dot{q}_i$ and $\lambda_0^n \rightarrow \lambda_0$ at time $t$ may be demonstrated by following the procedures and arguments presented for state holonomic systems.$^{18}$

**Adjoint dynamical system**

Having theoretically converged to the true accelerations and Lagrange multipliers of the state nonholonomic dynamical system at time $t$, we now focus on the equations defining the adjoint system. Recall the equations governing the adjoint system

$$\dot{\eta}_j = -v_i \left( \frac{\partial h_i}{\partial q_{ij}} + \frac{\partial g_i}{\partial q_{ij}} + \frac{\partial h_m}{\partial q_{ij}} u_m - \frac{\partial d_i}{\partial q_{ij}} \lambda_0 \right)$$

$$+ \gamma_0 \frac{\partial \phi_{ej}}{\partial q_{ij}} \dot{q}_j - \frac{d}{dt} (\gamma_0 \phi_{ej})$$

subject to $v_i d_i = 0$

One key observation regarding these equations, viz., Eqs. (11a), is they involve the first derivative of the Lagrange multipliers $\gamma_0$: this is a direct consequence of the (Pfaffian) form of the nonholonomic constraint equations and presents a problem for the augmented Lagrangian penalty method. To eliminate the appearance of $\gamma_0$, we transform the differential equations to first order form via

$$\dot{\eta}_j = -v_i \left( \frac{\partial h_i}{\partial q_{ij}} + \frac{\partial g_i}{\partial q_{ij}} + \frac{\partial h_m}{\partial q_{ij}} u_m - \frac{\partial d_i}{\partial q_{ij}} \lambda_0 \right)$$

$$+ Q_{ij} \dot{q}_j + \gamma_0 \frac{\partial \phi_{ej}}{\partial q_{ij}} \dot{q}_{ij}$$

Equations (11) then become

$$\dot{\eta}_j = \gamma_0 + v_i \frac{\partial h_i}{\partial q_{ij}} - Q_{ij} \dot{q}_{ij} + \gamma_0 \phi_{ej}$$

subject to $v_i d_i = 0$

Note that $\dot{\eta}_j$, $\dot{\eta}_j$, and $\gamma_0$ are needed to integrate the equations to the next time step. In company with this statement, it is important to notice the one-way coupling that the transformation has revealed within the adjoint system: the constraint equation involves the costates $v_i$ but not the costates $\eta_i$. This serendipitous coupling
allows us to apply the augmented Lagrangian method to a part of the adjoint system—namely Eqs. (21b,c), where we iterate until we converge upon \( \psi \) and \( \gamma \)—after which the entire adjoint system may be integrated forward in time. A side effect of this transformation is that the transversality conditions are slightly altered.\(^{18}\)

We begin the iterative process for the adjoint system by gathering terms in Eqs. (21b) and writing

\[
\dot{\psi}_j = Y_j(q, \dot{q}, v, \eta) - \phi_{ij} \gamma_0 \tag{22a}
\]

subject to \( \psi_0(q, v) \equiv v_i d_{io}(q) = 0 \) \( \tag{22b} \)

An approximation to the partitioned adjoint system is given by

\[
\dot{\psi}^n_j = Y_j(q, \dot{q}, v, \eta) - \phi_{ij} \gamma^n_0 - \frac{1}{\epsilon} \phi_{ij} \psi^n_o \tag{23a}
\]

\[
\gamma^{n+1} = \gamma^n_0 + \frac{1}{\epsilon} \psi^n_o \quad \text{with} \quad \gamma^0_0 = 0 \tag{23b}
\]

where

\[
\psi^n_o \equiv \psi_0 + \omega \psi_0 = \left[ d_{io} \dot{\psi}_i + v_i \frac{\partial d_{io}}{\partial q_k} \dot{q}_k + \omega v_i d_{io} \right] \]

To establish the convergence of the iterative process, instead of Eqs. (22), we consider an equivalent form given by

\[
\dot{\psi}_j = Y_j(q, \dot{q}, v, \eta) - \phi_{ij} \gamma_0 - \frac{1}{\epsilon} \phi_{ij} \psi_o \tag{24a}
\]

\[
\gamma = \gamma_0 + \frac{1}{\epsilon} \psi_o \tag{24b}
\]

where

\[
\psi_o \equiv \psi_0 + \omega \psi_0 = \left[ d_{io} \dot{\psi}_i + v_i \frac{\partial d_{io}}{\partial q_k} \dot{q}_k + \omega v_i d_{io} \right] \]

From this point, convergence of \( \dot{\psi}^n_j \rightarrow \dot{\psi}_j \) \( \gamma^n_0 \rightarrow \gamma_0 \) at time \( t \) may be demonstrated by subtracting Eqs. (24) from (23) and following the now familiar arguments.\(^{18}\)

**Implementation**

Thus the iterative process of the augmented Lagrangian method when applied to Pfaffian nonholonomic systems provides \( \dot{q}, \dot{v}, \lambda \) and \( \gamma \) at a particular time, and integration of the state and adjoint dynamical systems to the next time step may proceed. Implementation of this approach was discussed in the previous section, and applies, mutatis mutandis, to Pfaffian nonholonomic systems.

**Examples**

In this section we present several example problems to help clarify the ideas and demonstrate the utility of the formulations. The routine DNEQNF, available from the IMSL mathematics library,\(^{20}\) was used to implement the shooting method for solving the TPBVPs associated with the examples.

**Example 1: Holonomic system**

Our first example of a holonomic system is a particle constrained to move on a circular path. The generalized coordinates are \( x \) and \( y \), and the governing equations are

\[
\ddot{x} = -uy + 2x \lambda
\]

\[
\ddot{y} = ux + 2y \lambda
\]

subject to \( \varphi(x, y) = x^2 + y^2 - 1 = 0 \)

The performance index reads \( J = \int_{t_0}^{t_f} \left[ \frac{1}{2} u^2 \right] dt \) and leads to the following adjoint system

\[
\ddot{v}_x = v_y u + 2v_x \lambda + 2x \gamma
\]

\[
\ddot{v}_y = -v_x u + 2v_y \lambda + 2y \gamma
\]

subject to \( \varphi(x, y, v_x, v_y) = 2vx + 2vy = 0 \)

The differential optimality condition is given by

\[
\ddot{u} = -gv_x + zv_y
\]

We consider a ten second, rest-to-rest maneuver of the particle. Because the state system has only one degree of freedom, we may specify either \( x \) or \( y \) at \( t_0 \) and \( t_f \); the other is determined by the holonomic constraint. The same statement applies to the velocities \( \dot{x} \) and \( \dot{y} \). We consider \( x \) as the independent generalized coordinate, and note that a rest-to-rest maneuver implies zero velocity and control at the initial and final times. For initial and final positions we take \( x(t_0) = 1 \), and \( x(t_f) = -1 \).

Because the initial conditions on the state system are specified, the initial values of the costates and their first derivatives are unknown and must be determined from the numerical solution of the TPBVP. But the adjoint system also has only one degree of freedom, so either \( \psi_0 \) or \( \gamma_0 \) at time \( t_0 \) is found from the solution of the TPBVP, and the other is determined from the constraint. Similarly for their derivatives. For this particular maneuver, we find \( \psi_0(t_0) \) and \( \gamma_0(t_0) \) from the numerical solution of the TPBVP. Note that because the boundary conditions on the control are specified, \( \dot{u} \) is unknown at \( t_0 \) and it too must be determined from the numerical solution of the TPBVP. So the solution process for this maneuver is as follows:

1. For \( x(t_0), \dot{x}(t_0), \) and \( u(t_0) \) specified, determine \( \gamma(t_0), \gamma_0(t_0) \) from the constraint on the state system. Guess \( \psi_0(t_0), \psi_0(t_0) \) and \( \psi_0(t_0) \) from the constraint on the adjoint system.

2. Integrate forward, using the augmented Lagrangian method at each time step to enforce the constraints on the state and adjoint systems. The values of certain parameters in the method are listed in Table 1, where \( \epsilon^* \) is the value at which convergence of the iterative scheme is recognized.
3. At the final time $t_f$, compare the values of $x(t_f)$, $\dot{x}(t_f)$, and $u(t_f)$ with their specified values. Use the errors to determine the updated guessed values of $v_2(t_0)$, $\dot{v}_2(t_0)$, and $\dot{u}(t_2)$.

The numerical solution for this maneuver is presented in Fig. 1, where item a shows the trajectory of the particle as it moves from the position $(x, y) = (1, 0)$ to $(x, y) = (-1, 0)$, while item b shows the corresponding optimal control profile. All of the specified boundary conditions are seen to be satisfied.

In item c, we show the Lagrange multipliers that enforce the constraints on the dynamical systems: the Lagrange multipliers acting on the state and adjoint systems are denoted $\lambda_1$ and $\lambda_2$, respectively. The last item in the figure shows the 2-norm of the constraint functions $\Phi$ and $\Psi$. Item d evinces that each function is maintained below the convergence tolerance $\epsilon^*$ throughout the maneuver. We further comment that the symmetry of the solution is plainly seen in each item of this figure, even in the constraint profiles. From this simple example, it seems that our approach for optimal control of holonomic systems is valid and promising for numerical solutions.

**Example 2: Holonomic system**

Our next example models a rigid two-arm manipulator handling a rigid payload. The shoulder of each arm is fixed, and a diagram of the system is shown in Fig. 2. This system has seven generalized coordinates and four constraints, hence three degrees of freedom. The seven generalized coordinates are $q = [\theta_1, \theta_2, \theta_3, \theta_4, x, y, \alpha]'$. Here, each $\theta$ specifies the orientation of the respective link as measured from a fixed horizontal reference, and $\alpha$, also measured from a fixed horizontal reference, describes the angular orientation of the payload. All values are positive counterclockwise. The coordinates $x, y$ define the payload center of mass. The four constraints on the system arise because the grapple points of the two arms (essentially the end effector of each arm) are related to the position and attitude of the payload. We consider torque actuators at the shoulder and elbow of each arm, and furthermore, we assume a gravity-free environment so the system Lagrangian equals the kinetic energy.

The performance index reads $J = \frac{1}{2} \int_{t_0}^{t_f} \left[ \ddot{u}_1^2 + \ddot{u}_2^2 + \ddot{u}_3^2 + \ddot{u}_4^2 \right] dt$, and further details, including the governing DAEs and differential optimality conditions, are considered in reference 18.

We consider the payload coordinates to be the independent coordinates, and we take the initial and final positions of a ten second rest-to-rest maneuver to be

\[
\begin{bmatrix}
\alpha(t_0) & x(t_0) & y(t_0)
\end{bmatrix} \begin{bmatrix}
0^\circ & \frac{1}{2} + \sqrt{2} & 0
0^\circ & \frac{3}{2} + \sqrt{2} & 0
\end{bmatrix}
\]

The initial values of the left and right robot links are determined from the holonomic constraints. Also, recall that rest-to-rest maneuvers imply zero initial and final velocities and controls.

We consider the costates associated with the payload coordinates to be the independent costates, so the initial values of these costates and their first derivatives are found from the solution of the TPBVP, while the other costates and derivatives must be consistent with the constraints on the adjoint system at time $t_0$. Furthermore, because the boundary conditions on the controls are specified, the four control derivatives are unknown at $t_0$ and must be determined from the numerical solution of the TPBVP.

From this point, the solution process for this example is similar to the other. The model parameters for this example are listed in Table 2, and $L$, the distance between the two shoulders, equals $2\sqrt{2}+1$ meters. The parameter values for the augmented Lagrangian method are listed in Table 1.

The numerical solution of the ten second maneuver is presented in Fig. 3. Items a and b show the corresponding optimal control profiles of the left and right arms. We remark that all specified payload conditions are satisfied.

**Example 3: Pfaffian nonholonomic system**

A classic example of a nonholonomic system models a vertical wheel rolling without slip on a horizontal plane. Four coordinates are necessary to describe the configuration of the wheel, and we select the contact point of the wheel with the surface, the rotation (or rolling) angle, and the heading angle. These coordinates are expressed by the variables $q = [x \ y \ \alpha \ \beta]'$. The heading angle $\beta$ is measured from the fixed $z$ axis, with positive values taken counterclockwise.

There is no potential energy associated with the model, so the system Lagrangian equals the kinetic energy. Furthermore, we consider driving and steering control inputs, given by $u_1$ and $u_2$, respectively. The ‘roll without slip’ condition is described by the two nonintegrable constraints

\[
\dot{x} = r \dot{\alpha} \cos \beta, \quad \dot{y} = r \dot{\alpha} \sin \beta
\]

where $r$ is the wheel radius. It is evident that these constraints are of the classical Pfaffian form. For simplicity, all model parameters are taken as unity.

The performance index reads $J = \int_{t_0}^{t_f} \left[ \frac{1}{2} \dot{u}_1^2 + \frac{1}{2} \dot{u}_2^2 \right] dt$, and further details, including the governing DAEs and differential optimality conditions are considered in reference 18.

We consider a ten second, rest-to-rest maneuver of the wheel. At the initial and final times, the nonholonomic constraints allow us to specify only two generalized velocities, while the other velocities must be consistent with the constraints. The four generalized positions, however, are not restricted. For this maneuver
we select \( \dot{\alpha} \) and \( \dot{\beta} \) to equal zero at \( t_0 \) and \( t_f \)—rest-to-rest maneuvers imply zero velocity and control at the initial and final times. Our initial and final positions are specified as

\[
\begin{align*}
q(t_0) &= [0 \ 0 \ 0]'
\end{align*}
\]

\[
\begin{align*}
q(t_f) &= [2\pi \ 2\pi \ \text{free} \ \pi/2]'
\end{align*}
\]

where the entry for \( \alpha(t_f) \) specified as 'free' means that we do not specify the final value. This maneuver describes a 90° turn in heading together with placement of the wheel at a point, while the final rotation angle is unimportant.

The costates corresponding to position, \( \eta_i \), are not constrained, so each is determined from the solution of the TPBVP. The costates corresponding to velocity, however, are constrained. We consider the costates associated with \( \dot{\alpha} \) and \( \dot{\beta} \) as the independent costates, so their initial values are found from the solution of the TPBVP, while the costates associated with \( \dot{\alpha} \) and \( \dot{\beta} \) at time \( t_0 \) are determined from the constraints. And finally, because the boundary conditions on the control are specified, the two control derivatives are unknown at \( t_0 \) and must be determined from the numerical solution of the TPBVP.

The numerical solution for the ten second maneuver is presented in Fig. 4. Item a shows the contact point trajectory of the wheel as it rolls from the origin to the point \((x, y) = (2\pi, 2\pi)\) in the horizontal plane. The wheel completes one and one-half revolutions during the maneuver. Item b displays the driving and steering control input profiles. For this rest-to-rest maneuver, the driving control, \( u_1 \), has a familiar acceleration and deceleration shape. From this simple example, it seems that our approach for optimal control of Pfaffian nonholonomic systems is valid and promising for numerical solutions.

Conclusions

In this paper, we have presented new methods in optimal control for holonomic and Pfaffian nonholonomic systems. Our approach involves appending the differential equations of motion and algebraic equations of constraint to a performance index. The necessary conditions for optimal control form a two-point boundary-value problem, described by a one-way coupled set of differential-algebraic equations. The special one-way coupling is exploited in using an augmented Lagrangian penalty method for solving the time and space evolution of the differential-algebraic equations. The development of the unique form of these necessary conditions and the extension of the augmented Lagrangian penalty method from initial value problems involving differential-algebraic equations to two-point boundary-value problems involving differential-algebraic equations constitutes the main contributions of this paper. We have included several example problems to help clarify the ideas and demonstrate the utility of the formulations.

References

14. Bloch, A.M. and McClamroch, N.H., "Control of Mechanical Systems with Classical Nonholonomic Con-
Table 1. Augmented Lagrangian method parameters for Examples 1 & 2.

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>$1/\sqrt{2}$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\sqrt{2}/dt$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\epsilon^*$</td>
<td>1.0E-15</td>
</tr>
</tbody>
</table>

Table 2. Model parameters for Example 2.

<table>
<thead>
<tr>
<th>Body $i$</th>
<th>$m$, kg</th>
<th>$l$, m</th>
<th>$I$, kg-m$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>12.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>12.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>4</td>
<td>12.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>p</td>
<td>6.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

generalized coordinates: 7  
controls: 4  
constraints: 4  
degrees of freedom: 3  

Figure 2. Rigid two-arm manipulator and payload.
Figure 1. Optimal control solution for Example 1.

Figure 3. Optimal control solution for Example 2.

Figure 4. Optimal control solution for Example 3.
generalized coordinates: 7
controls: 4
constraints: 4
degrees of freedom: 3

Figure 2. Rigid two-arm manipulator and payload.