Robust Control of Redundantly Actuated Dynamical Systems

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The eigenstructure assignment scheme for robust multivariable state feedback control is extended to redundantly actuated dynamical systems. The fact that the plants that are at least square allow for the exact assignment of an arbitrary orthonormal set of closed loop eigenvectors makes this methodology attractive for feedback controller design poorly modeled dynamical systems. For second order mechanical systems, a partial eigenstructure assignment methodology is presented which simplifies the scheme of pole assignment for real target sub-modal matrix. An optimal allocation strategy is derived for redundant control inputs to minimize an integral cost function. A novel adaptive aggregation procedure is introduced to handle problems with very high redundancy in actuation. The aggregation approach along with the eigenstructure assignment methodology is applied to the wing morphing problem. The robust redundant actuation procedure is demonstrated on a mathematical model constructed from the wind tunnel test data of the novel morphing wing model designed and developed at Texas A & M University.

Nomenclature

\begin{itemize}
  \item \(A\) System Matrix of a linear dynamical system
  \item \(B\) Control Influence Matrix
  \item \(x\) Vector of State Variables
  \item \(u\) Vector of Control Variables
  \item \(M\) Mass matrix of a second order mechanical system
  \item \(C\) Damping matrix of a second order mechanical system
  \item \(K\) Stiffness matrix of a second order mechanical system
  \item \(q\) Vector of generalized coordinates
  \item \(\sigma(\cdot)\) Singular Value of \((\cdot)\)
  \item \(\|\cdot\|\) Frobenius norm of \((\cdot)\)
  \item \(\eta\) Vector of modal coordinates
  \item \(i\) Variable number
\end{itemize}

I. Introduction

Redundancy in sensing and actuation has taken center stage in control of modern day dynamical systems owing to their complex functionality and increased performance targets. Classical control theory being developed for under-actuated systems, with the fundamental considerations of stability in regulation or tracking. The advent of redundancy enables the engineer to design for increased goals of system performance such as robustness (to model errors and other perturbations), coordination (of redundant actuator sets) and reliability enhancement (in case of actuator failure). This paper proposes robust solutions and implementation methods for the actuator redundancy problem at several levels. Let us first look at some applications in lieu of the emerging importance of redundantly actuated dynamical systems.

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In robotic systems, redundancy is handled by several strategies. One particular approach shown in Nakamura uses a prioritization approach achieving local/global optimal control formulations. Typical applications include robots with parallel actuated wrists and dexterous fingers. Modeling of such closed link robot mechanisms with redundant actuators poses an equally difficult challenge. The advantages of redundancy in handling actuator failure and in optimizing load distribution were pointed out by Nakamura and Ghodoussi. They suggested modeling of redundant joints by equivalent open chains and control optimization was done by a simple optimization scheme. A simpler dynamics methodology with a comparison of several redundant control schemes was proposed by Cheng. A computationally fast methodology for multi-fingered grasping applications, Dynamics Force Distribution, is presented by Zheng. Junkins et al. propose a novel methodology for coordination of redundant sets of actuators in multibody systems with a weighted pseudoinverse, with weights determined by the current work space coordinates.

In aircraft control problems, with the on-set of non-traditional control surfaces, the redundant actuation methods have taken a center stage, in recent times, with increase in computation power and hardware capacities. Prioritization of redundant actuator sets is one of the methods of handling redundancy in aircraft control, such schemes, called “Daisy Chain Control Allocation” were first demonstrated by Buffington. When the redundant sets of actuators are actually constrained by their operation limits, Durham proposes constrained control allocation schemes. Bodson presents an overview of constrained control allocation strategies. In the presence of actuator constraints, the pseudo-inverse solution cannot meet the demanded accelerations computed from a square plant. In the light of the same, direct allocation, which Bodson poses as a linear programming problem, presents a highly desirable candidate for real-time application. Singla proposes an approximation approach to recursively allocate using adaptive distribution functions addressing the high dimensionality of the control space with computational efficiency with an application to the morphing problem.

In spacecraft attitude control problems, the Control Moment Gyros (CMG) and Reaction Wheels form a major class of actuators. Four Single Gimbal CMGs (SGCMGs) in a pyramidal configuration, form a redundant set of actuators in the reorientation maneuvers of spacecraft. Design of a steering law for redundant SGCMGs was modified by Bedrossian to avoid such singularities. Schaub, et.al. extended this methodology to obtained control laws for variable speed CMGs for a better redistribution of the internal momentum.

Organization

The paper is organized as follows. Section II presents the robust eigenstructure assignment algorithm with application to redundantly actuated dynamical systems. The method’s relevance originates from the fact that any orthogonal closed loop target is always exactly assignable when the plant is atleast square. This scheme is then extended to second order systems via a simple technique called the partial eigenstructure assignment. The allocation of thus calculated robust control among redundant actuator sets is carried out by a control distribution scheme posed as an optimal control problem. The open loop version of this scheme is demonstrated on an empirical model of a novel morphing wing designed and developed at Texas A&M University. For hyper-redundant actuation (required for application in smart actuator arrays), an adaptive aggregation scheme is developed to group actuators in to finite groups so that the reduced dimension actuator sets become amenable for allocation methods. Some conclusions and future directions are presented later.

II. Robust Eigenstructure Assignment

Eigenstructure assignment is an established tool for robust control of state-space models by state and output feedback. The flexibility of the methodology to allow the designer to explore the arbitrariness of the closed loop admissible eigenvectors in addition to the pole assignment made the method a popular tool for feedback gain calculations. It is well known that near orthogonality of the closed loop eigenvectors is desirable to minimize sensitivity of the eigenvalue placement to the model errors. Arbitrary selection of the closed loop eigenvectors can lead to a highly sensitive gain matrix, making the conditioning of the closed loop modal matrix a key issue in the achievement of robustness of the controller to plant parametric uncertainties. The design process can be summarized as follows.
II.A. Eigenstructure Assignment

Consider the following state space description of a linear dynamical system

\[ \dot{x} = Ax + Bu \]  
\[ x(t_0) = x_0 \]

with a state feedback control law, \( u = -Gx \) where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{m \times n} \). resulting in a closed loop of the form,

\[ \dot{x} = (A - BG)x \]

The corresponding closed loop eigenvalue problems can be written as

\[ (A - BG)\phi_i = \lambda_i \phi_i \]
\[ (A - BG)^T \psi_i = \lambda_i \psi_i \]

with the left and right eigenvectors \( \phi_i, \psi_i \) being normalized according to the conditions

\[ \phi_i^T \psi_i = 1 \]
\[ \psi_i^T \phi_j = \delta_{ij} \]

By introducing a parameter vector, \( h_i = G\phi_i \), the problem of calculation of the gain matrix, \( G \), becomes the solution of the matrix Sylvester equation,

\[ A\Phi - \Phi\Lambda = BH \]

where \( \Phi = [\phi_1, \phi_2, ..., \phi_n], \Lambda = \text{diag}[\lambda_1, \lambda_2, ..., \lambda_n] \) and \( H = [h_1, h_2, ..., h_n] = G\Phi \). For a given target modal matrix, \( \Phi \), we would need to solve for the gain matrix, \( G \), that would bring about the assignment of the required closed loop eigenvectors. The robustness of this scheme is derived from a theorem due to Patel and Toda.\(^3\)

We state the theorem and discuss the robustness properties.

**Theorem II.1 (Patel and Toda\(^3\))** Consider the asymptotically stable closed loop system \( \dot{x} = A_{CL}x \). The perturbed closed loop system matrix, \( A_{CL} + \Delta \) of the original system remains asymptotically stable for all perturbations, \( \Delta \), bounded by

\[ \|\Delta\| < \frac{\min\{-\Re\{\lambda(A_{CL})}\}}{\kappa(\Phi_{CL})} \]

where, \( \Phi_{CL} \) is the closed loop modal matrix, \( \kappa(.) \) is the condition number defined as the ratio of the maximum to the minimum singular values of the matrix, \( \lambda(.) \) and \( \Re\{\} \), denoting the real part of \( \{\} \).

**Proof** The proof of this theorem is given by Patel and Toda\(^3\) and by Junkins and Kim\(^1\) using Lyapunov theory. This theorem is the manifestation of the intimate connection between the conditioning of the algebraic eigenvalue problem and the robustness stability margin for a linear dynamical system subject to additive plant parameter perturbations.

Therefore, in principle, for a square plant (a plant model with at least as many actuators as the number of state variables, \( m \geq n \)), any additive parametric perturbation can be compensated by state feedback via eigenstructure assignment. This application to redundantly actuated dynamical systems is summarized by the following theorem.

**Theorem II.2** A redundantly actuated linear dynamical system,

\[ \dot{x}(t) = Ax + Bu \]

controlled by a linear full state feedback control law, \( u = -Gx \), \( u \in \mathbb{R}^m, m \geq n \), \( B \) full row rank, with \( G = B^\dagger(A\Phi - \Phi\Lambda_{CL}) \), where \( \Phi \) is a target orthonormal set of closed loop eigenvectors and the Hurwitz target closed loop eigenvalue matrix, \( \Lambda_{CL} \), chosen such that, \( \min\{-\Re\{\lambda_{CL}(i)\}\} > \|\Delta\|_2 \), ensures the robust stabilization of the linear system to all perturbations bounded in magnitude by the matrix norm \( \|\Delta\|_2 \).
Proof The proof follows from theorem 1. Because, \( m \geq n \), atleast one gain matrix exists to solve the Sylvester equation exactly as shown below (\( n^2 \) equations in \( mn \) variables).

\[
(A\Phi - \Phi\Lambda_{CL})\Phi^{-1} = BG
\] (9)

Further, the choice of an orthonormal matrix for closed loop modal matrix, \( \Phi \), together with Patel Toda theorem II.1, ensure the closed loop poles to satisfy the following bound (with the additional fact of perfect conditioning of the closed loop modal matrix, \( \kappa(\Phi) = 1 \)).

\[
\|\Delta\|_2 < \min\{-\Re\{\lambda_{CL}^{(i)}\}\}
\] (11)

This ensures the robust stabilization.

\[\square\]

II.B. Second Order Mechanical Systems: Partial Eigenstructure Assignment

In mechanical systems, often, it is desired to construct very high order plant models (For Eg., flexible structures). Also, in most mechanical system models, the kinematics remain unforced, thus leading to a cascaded system of differential equations. Therefore, it is important to understand how the above results for redundant actuation translate to the second order mechanical systems. Accordingly, a classic paper by Hughes and Skelton,\(^4\) specializes the known results for first order state space models to second order models, important for mechanical systems. This motivated many control theorists to search for a second order extension to eigenstructure assignment scheme. Juang et. al.,\(^5\) propose a Eigenstructure assignment for second order systems by an IMSC method. Juang and Maghami\(^6\) extended this, by incorporating a sequential scheme that assigns the closed loop eigenvectors using subspace intersection between the desired closed loop eigenspace and the assignable subspace of eigenvectors. Kim et. al.,\(^7\) propose a computationally efficient method of achieving the same.

In the same spirit, we present a partial eigensystem assignment algorithm for second order mechanical systems. Apart from its elegance, this methodology is computationally very attractive, especially for high order models, where the real time compuation of feedback gains is an important problem and it is desirable to solve as simple an algebraic problem as possible.

Most of the second order mechanical systems, in practice allow the assignability of a part of the eigenstructure. This is because, the external forces do not apply directly on the kinematics part of the dynamical system. It therefore follows from the developments of Srinathkumar,\(^8\) that only \( n \) components (of the total \( 2n \) state variables, ofcourse assuming there are atleast \( n \)) of each mode can be assigned arbitrarily. This is sufficient for most practical applications, if we desire a complex conjugate pair of target set of closed loop eigenvalues. This is because of the following structure of the modal matrix for any linear second order mechanical system.

\[
\Theta = \begin{pmatrix}
\Phi & \Phi \\
\Phi\Lambda & \Phi\Lambda
\end{pmatrix}
\] (12)

Clearly, because of the constraints from the kinematics and the complex conjugate eigenvalues of the second order system, the determination of the \( n \times n \) principal sub modal matrix, \( \Phi \), determines the entire modal matrix. We utilize this fact and try and make the principal sub modal matrix, \( \Phi \), as well conditioned as possible. Now we develop a systematic method to do the partial EsA.

Let the second order mechanical system be given by

\[
M\ddot{x} + C\dot{x} + Kx = Bu
\] (13)

with \( M \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{m}, B \in \mathbb{R}^{m \times n} \), and \( m \geq n \), \( \text{rank}(B) = n \). With a position and velocity feedback, \( u = Gx = G_1x + G_2\dot{x} \), the closed loop eigen value problem for the second order mechanical system is given by

\[
\left(M\lambda_i^2 + (C - BG_2)\lambda_i + (K - BG_1)\right)\phi_i = 0
\] (14)
where, $\lambda_i \in \mathbb{C}^n = \lambda_{Ri} + j\lambda_{Ii}, \lambda_{Ri}, \lambda_{Ii} \in \mathbb{R}$ and $j^2 = -1$. We assign a set of distinct complex target poles and their conjugates are assigned automatically. Consider a real $n \times n$ principal modal sub-matrix. A unitary target for sub modal matrix makes the problem more complicated, with coupled matrix equations to solve for the required feedback gain matrices. For simplicity and elegance, we consider the real sub-matrix in this paper. Separating the real and imaginary parts of the above equation, we get the following two conditions,

$$
M(\lambda_{Ri}^2 - \lambda_{Ii}^2) + (C - BG_2)\lambda_{Ri} + (K - BG_1) \phi_i = 0 \quad (15)
$$

$$
j(2M\lambda_{Ri}\lambda_{Ii} + (C - BG_2)\lambda_{Ii}) \phi_i = 0 \quad (16)
$$

Clearly the equations above represent a cascade and one of them is the matrix Sylvester equation, we saw earlier. From the above conditions, together with the fact that we can assign a target $n \times n$ principal modal sub matrix exactly owing to the fact that the plant is atleast square, we have the following feedback gains.

$$
G_2 = B^{-1}(C + 2M\Phi\Lambda_R\Phi^T) \quad (17)
$$

$$
G_1 = B^{-1}(M\Phi(\Lambda_R^2 - \Lambda_I^2)\Phi^T - 2M\Phi\Lambda_R^2\Phi^T + K) \quad (18)
$$

It is found that these gains assign the closed loop poles along with the principal target modal sub-matrix exactly (subject to appropriate scaling and complex magnitude considerations). We demonstrate the same with the following example problem (example 5.4 from Junkins and Kim$^1$) and discuss the positive features of the algorithm proposed above.

**II.B.1. Example**

Consider the undamped open loop second order system with the following parameters,

$$
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, K = \begin{pmatrix}
20 & -10 & 0 \\
-10 & 30 & -20 \\
0 & -20 & 20
\end{pmatrix} \quad (19)
$$

With such parameter matrices, consider the target eigenstructure given by $\Lambda_{target} = diag(-0.72 + 1.24i, -0.72 - 1.24i, -2.23 + 3.87i, -2.23 - 3.87i, -3.4 + 5.9i, -3.4 - 5.9i)$, that is $\omega_1 = 1.44$, $\omega_2 = 4.47$, $\omega_3 = 6.92$, $\zeta_1 = \zeta_2 = \zeta_3 = 0.5$. The assigned eigenvalues computed from the above methodology and the corresponding closed loop eigenvectors are shown below.

$$
\Lambda = diag(-0.72 + 1.24i, -0.72 - 1.2i, -2.2 + 3.8i, -2.2 - 3.8i, -3.46 + 5.9i, -3.46 - 5.9i), \quad (20)
$$

$$
\Phi = \begin{pmatrix}
-0.52218 - 0.533i & 0 & 0 \\
0 & -0.11656 - 0.1935i & 0 \\
0 & 0 & -0.0737 - 0.125i \\
1.0405 - 0.0267i & 0 & 0 \\
0 & 1.012 - 0.022i & 0 \\
0 & 0 & 1.0051 - 0.00918i
\end{pmatrix} \quad (21)
$$

The above equations are the representative modes of closed loop system. The other three modes are the complex conjugates of the above modes. At first blush, it may seem that this is not the eigenstructure we wanted. That is due to the bi-orthogonality conditions we imposed for normalization. Upon dividing each column so as to make the principal sub-modal matrix unity, we unravel the following scaled modes (That is divide ith eigenvector with its $i, ith$ entry, in the above set).

$$
\Phi_{scaled} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-0.72 + 1.2471i & 0 & 0 \\
0 & -2.235 + 3.8711i & 0 \\
0 & 0 & -3.46 + 5.9929i
\end{pmatrix} \quad (22)
$$

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Clearly, this was the target eigenstructure. We also recover the condition number by this scaling to have the same conditioning as the target modal matrix. This can be compared with the solution by projection method in Junkins and Kim\(^1\) in table 5.3. Comparing with the projection method, using a unitary matrix as a target, the norm of the feedback gains required was \(\|G\|_F = 50.91\), where as in this case, the norm of the gain matrix was found to be \(\|G\|_F = 37.363\).

Thus we can see from the above that we need not always use a projection method to stay as close as possible to a known, well conditioned system. It is also of consequence if we concentrate on the assignable sub modal matrix as the second order mechanical systems have a special structure and with a square plant, we can always assign the assignable target sub matrix.

### II.B.2. Features of the Partial EsA algorithm

Whenever possible, the partial EsA algorithm has the following advantages and disadvantages when compared to its cousins proposed in Juang and Maghami,\(^6\) Juang et.al.,\(^5\) and Kim, et.al.,\(^7\)

- The algorithm above is computationally lesser expensive than its cousins. Juang\(^5\) algorithm uses SVD for each eigenvalue/vector assignment and a matrix inverse is required, whereas we need only one matrix inverse. Juang\(^6\)’s method requires a QR step and an SVD for each eigenvalue/vector as the procedure uses angle between subspaces. Kim’s approach is actually quite similar to the present approach, but does not use the fine matrix manipulations as the above algorithm.

- This algorithm doesn’t need the modal damping assumption which is important for Kim’s algorithm. The damping can be asymmetric. In fact, the only restriction we need in the above development is to have an invertible (or pseudo-invertible) control influence matrix and a positive definite mass matrix.

- The limitation of this method is that we need at least \(n\) actuators for \(2n\) state variables in the second order form. This can be relaxed but we cannot assign the target submodal matrix exactly. This would entail us to use a projection criterion to minimize the error between assignable vectors and ideal target vectors.

- The method needs an extension to assign complex principal modal sub matrices. This brings about a change in the structure of the conditions that need to be satisfied.

- An arbitrary selection of target eigenvector set can lead to very high gains and a sensitive feedback controller. Therefore we need to select the target modal matrices judiciously.

Therefore, eigenstructure assignment methodology forms an important class of robust control design strategies. Notice that we have not thusfar, specialized the method for redundantly actuated dynamical systems. The computation of control input to individual actuators is in turn done by a minimum norm solution. In the next section we present a novel alternative scheme to distribute this control input among the individual actuators using an optimality criterion.

### III. Redundant Control Allocation

Control Allocation problem focuses on allocating the control authority to individual inputs of the redundant actuator sets, so as to produce a desired total force/moment specified by a control law (equal in dimension to the state vector). The allocation process, in general, does not constitute the control law. The allocation of authority becomes especially important in the presence of control variable inequality constraints, as clipping the pseudo-inverse solution to satisfy the constraints leads to improper usage of redundancy. An excellent survey of existing methods to solve this problem is given by Bodson.\(^9\) Durham’s\(^10\) Direct allocation along with several other optimization criteria such as the error minimization and control minimization are discussed and associated algorithms are presented.

In this section, we propose a slightly different formulation to the allocation problem with certain desirable features. Consider the square reference model and the plant with redundant inputs given by

\[
\dot{y}_m = A y_m + a_d \tag{23}
\]

\[
\dot{y} = A y + B u \tag{24}
\]
where $u \in \mathbb{R}^n$, $a_d$, $y$, $y_m \in \mathbb{R}^m$, $m > n$. $a_d$ is the control law computed for the square plant model (in our case using a robust eigensystem assignment statefeedback scheme). Define tracking error $e \equiv y - y_m$ leading to the error dynamics, $\dot{e} = A e + B u - a_d$. The following allocation scheme minimizes an integral cost function

$$\min J = e(t_f)^T S_f e(t_f) + \frac{1}{2} \int_{t_0}^{t_f} u^T R u + (B u - a_d)^T \Gamma (B u - a_d) + e^T Q e \, dt$$

subject to

$$\dot{x} = A x + B u$$

$$u_{min} \leq u(t) \leq u_{max}$$

Hamiltonian can be written as

$$H(x, u, \lambda, \mu_1, \mu_2, t) = \frac{1}{2} (u^T R u + (B u - a_d)^T \Gamma (B u - a_d) + e^T Q e) + \lambda^T (A x + B u) + \mu_1^T (u - u_{max}) + \mu_2^T (u_{min} - u)$$

The necessary conditions for optimality are given by

$$H_u = R u + B^T \Gamma B u - B^T \Gamma a_d + B^T \lambda + \mu_1 - \mu_2 = 0$$

$$\dot{\lambda} = -Q e - A^T \lambda$$

$$\dot{x} = A x + B u$$

with boundary conditions $x(t_0) = x_0, \lambda(t_f) = S_f e(t_f)$.

**Remarks** In the above formulation, the integral cost function penalizes the deviations from the desired control inputs $a_d$, keeping the inputs $u$ finite and ensuring the boundedness of the state variables. Guesses for active times of the inequality constraints and an initial guess for $\lambda(t_0)$, that ensures the convergence at $t_f$ is required.

On the other hand, if it is known that the control variable inequality constraints do not become active within the interval $[t_0, t_f]$, a feedback solution can be implemented which involves the solution similar to a linear quadratic tracking controller. This is desirable to avoid the determination of initial guesses. The feedback solution, in the unconstrained case is given by $u(t) = (R + B^T \Gamma B)^{-1} B^T (\Gamma a_d - \lambda)$ (Eq. 29 with $\mu_1, \mu_2 = 0$), where, $\lambda(t) = S(t) e(t) + v(t)$ where the gains $S(t), v(t)$ are calculated backwards in time as a solution to the matrix differential equations

$$\dot{S} + A^T S + SA - SB(R + B^T \Gamma B)^{-1} B^T S = Q$$

$$\dot{v} + A^T v - SB(R + B^T \Gamma B)^{-1} B^T v = S(B(R + B^T \Gamma B)^{-1} B^T \Gamma) a_d - Q y_m$$

with boundary conditions $S(t_f) = S_f$, $v(t_f) = -S_f y_m$.

**III.A. Example: Application to the Morphing Wing**

The open loop strategy of control allocation is applied to an empirical model of a novel morphing wing. The wing, shown in Fig. 7, consists of three twisting sections that change the wing twist profile, owing to the elastomeric skin that deforms on a flexible skeleton. Considering the wing as a control surface producing only Lift force and Rolling moments, it becomes a redundantly actuated system with the twist rates and angle of attack rate as control variables. From these considerations, an approximate model for the rate of change of lift and moment, for a steady potential flow (Assuming the rates are sufficiently slow to neglect the unsteady effects) would be

$$\frac{dC_L}{dt} = \frac{\partial C_L}{\partial \alpha} \dot{\alpha} + \frac{\partial C_L}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial C_L}{\partial \theta_2} \dot{\theta}_2 + \frac{\partial C_L}{\partial \theta_3} \dot{\theta}_3$$

$$\frac{dC_M}{dt} = \frac{\partial C_M}{\partial \alpha} \dot{\alpha} + \frac{\partial C_M}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial C_M}{\partial \theta_2} \dot{\theta}_2 + \frac{\partial C_M}{\partial \theta_3} \dot{\theta}_3$$

where $C_L, C_M$ are the lift and rolling moment coefficients and $\alpha, \theta_1, \theta_2, \theta_3$, being the angle of attack and twist angles at first, second and tip twisting sections respectively.
Static wind tunnel tests for the wing at several configurations were performed in the low speed wind tunnel at Texas A & M University. From the test data, the sensitivities for each input were calculated and thus we have the following model for the system above. Therefore the model is valid only for sufficiently small rates of change of angle of attach and the twist angles.

\[
\begin{bmatrix}
    \dot{C}_L \\
    \dot{C}_M
\end{bmatrix}
= \begin{bmatrix}
    0.0836 & 0.041 & 0.0255 & 0.0105 \\
    -0.0499 & -0.0281 & -0.015 & -0.0105
\end{bmatrix}
\begin{bmatrix}
    \dot{\alpha} \\
    \dot{\theta}_1 \\
    \dot{\theta}_2 \\
    \dot{\theta}_3
\end{bmatrix}
\]  \tag{36}

Consider, for simplicity, the unconstrained solution for this problem. Let the state feedback gains be calculated by an Eigenstructure assignment methodology assuming two inputs and a control effectiveness matrix to be identity matrix as shown

\[ a_d = -G[C_L C_M]^T \]  \tag{37}

\[ G = (A\Phi - \Phi A_{CL})\Phi^{-1} \]  \tag{38}

where for the problem of interest \( A = 0_{2 \times 2} \). Consider a solution to this problem using the open loop approach discussed in this section denoted by \( u_{\text{redundant}} \). Let us compare this solution with a solution obtained by pseudoinverse denoted by \( u_{\text{pseudoinverse}} = B^T(BB^T)^{-1}a_d \). These solutions are presented for a \( t_f = 10 \) seconds in the following figures.

![Figure 1. State Trajectories using Optimal Allocation procedure: Open Loop Approach](image)

Figure 1 shows the error dynamics \( e(t) \) along with the state trajectories of the redundantly actuated plant \( y(t) \) and its square reference model \( y_m(t) \). Because of the terminal cost \( O(||S_f||) = 1e6 \), the error goes to a small number at \( t_f \). There are no large departures from the reference model as specified by the integral cost function. Perfect asymptotic tracking is not always possible as we are also minimizing the total energy input \( u_{\text{redundant}} \), although this can be specified by changing the weights or posing the problem in a different manner. The merit of the methodology in asymptotic tracking needs further investigation. In practical applications, for a small initial time, one can use this methodology and then switch to pseudoinverse solution after redundant input values become sufficiently small to avoid large initial transience due to state feedback.

Figure 2 compares the redundant input values from the above methodology (subscript \( \text{red} \)) and the classical pseudoinverse solution (subscript \( \text{pseudo} \)). Note many desirable features of this solution. First, the control appears to be devoid of large rates. This is important for our application as accelerating \( \theta_i \)’s cause unsteady
effects and violate the modelling assumptions. The method yields a rate which has very small acceleration, this is highly desirable for the application at hand. Then, the magnitude is much lower in the initial phase when the pseudoinverse solution yields a large magnitude. Figure 3 also shows the same. Also, the method minimizes the allocation error. Therefore, as indicated previously, this allocation method can be used to retain stability in the initial high magnitude transience and then one can switch back to the pseudo inverse solution.

The allocation problem has been posed as an optimal control problem desiring a solution in the neighbourhood of a square plant reference and minimizing the control effort and allocation error. These allocation procedures however become computationally difficult when the dimensionality of the actuators becomes very large. So, for a smart wing distributedly actuated by smart materials or synthetic jets, we need some order reduction methods to aggregate actuators of similar effectiveness. A strategy for adaptive aggregation is presented in the next section.

IV. Hyper-redundant Actuation: Dimensionality Reduction by Adaptive Aggregation

In smart structures, the actuation is distributed in space. Shape memory alloys and piezo-electric materials are examples for this type of actuators. Nominally, they have inputs at several places and the smart matter delivers a distributed force input to the structures they are installed upon, in accordance with the input voltage signal at that location. It is envisioned that there will be novel smart sensing and actuation technologies using nano-materials in the near future which will increase the input dimensionality by an order of magnitude more than the current smart systems. Such actuators are poorly modeled and the overall system is thereby poorly captured by the physics based models. Hence there is a need for the control synthesis methodology to be insensitive to parametric uncertainties. Stability of such systems which have a continuum of actuators, governed by partial differential equations is a highly developed area. Wang\cite{11,12} pioneered the extension of Lyapunov’s direct method to analyze the stability of such systems and is accredited to have developed several models of distributed parameter systems for optimization of aeroelastic systems.\cite{13} This later laid foundation for the development of distributed parameter control theory. Bailey and Hubbard\cite{14} were among the first to demonstrate the active vibration damping of a cantilever beam using distributed parameter control logic, via Lyapunov’s second method. This was extended to accommodate more general boundary conditions and non-uniform actuator spatial distributions by Burke and Hubbard.\cite{15} The interesting aspect of this approach of treating the control as a distributed parameter is that there is no need for the discretizing the states to obtain a discretized model of the distributed parameter plant. The control

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Actuator Inputs to Redundant actuators, Allocation Error}
\end{figure}
Control Magnitude Comparison by different methods – Unconstrained Case

\[ ||u_{\text{red}}||, ||a_d||, ||u_{\text{pseudo}}|| \]

Figure 3. Comparison of total control effort with pseudoinverse solution

A partial differential equation is obtained directly along with its own boundary conditions. However, the grace of this method is lost immediately owing to the fact that most of the real-life applications use the discretized equations due to the collocated nature of the sensor outputs and actuator inputs (although smart material may be represented as a continuum, the input voltages have to be applied at finite locations). Therefore to control smart materials, we propose an adaptive aggregation scheme.

IV.A. Algorithm for Adaptive Discretization

In the adaptive aggregation scheme, using a physics-based model, we populate the control influence matrix iteratively in accordance with a performance measure. This is done as described in figure 4. We start with a control influence matrix and evaluate the actuator position effectiveness by an actuator placement effectiveness measure. In the cantilever beam example presented in figures 4, 5, 6, we use the measure proposed by Hamdan and Nayfeh. This measure forms a matrix whose \( i,j \)th element approximates the influence of \( j \)th actuator on the \( i \)th mode of the system. The algorithm is shown to place actuator nodes at several locations along the length on a beam problem. The node insertion for each iteration is plotted in Fig. 5. Fig. 6 shows the improvement of condition number of the control influence matrix of the beam problem with each node insertion. However, the node insertion shown in this simple example is not sufficient. The node insertion at each iteration is performed only at locations with largest controllability index. This results in what is called a greedy algorithm leading to insertion of nodes at the same locations each time. As a result there are an increasingly large number of nodes at sensitive locations. This issue is to be handled by a more uniform methodology of node insertion. This methodology becomes more important for distributed actuation in large structures as there is an increased need for sensitivity-based aggregation in such applications. However, the practicality of the algorithm dictates to have it working along with a spatial discretization tool.

IV.B. Application to the Morphing Wing

The above discretization procedure is applied to a finite element model of the morphing wing introduced earlier. An overly simplified model, with large model errors is being used to demonstrate the methodology. The actual structure of the wing is nonlinear due to several elements. Viscoelasticity, together with the coupled aerodynamics make the problem more complex. As a preliminary analysis, the wing is approximated by three members in torsion with varying geometric parameters (polar moment of inertia). The mass and stiffness matrices are computed from a three element finite element model of this structure. Five different force distributions are used as control inputs on the three elements and a second order mechanical system is constructed from the same, with four state variables and five control inputs. The open loop eigenvalues
and close loop eigenvalues achieved are shown in Table 1, together with the condition numbers of the closed loop and the open loop modal matrix.

Clearly, we need a methodology to identify the unmodeled dynamics to compensate for the modeling errors that we have committed to this model. An implementation of this algorithm therefore should use an estimator of the eigen frequencies of the model error and try to compensate them to maintain stability and achieve the set performance goals. The Eigensystem Realization Algorithm is a potential candidate to extract the participating modes.

Thus, for this type of problems of hyper-redundant actuator arrays, which are poorly modeled in general, the discretization scheme, together with the robust redundant eigenstructure assignment is promising to be a simple and practical approach.

<table>
<thead>
<tr>
<th>Openloop Eigenvalues</th>
<th>Target Eigenvalues</th>
<th>Closeloop Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>0+115.5i</td>
<td>-100+115.5i</td>
<td>-100+115.5i</td>
</tr>
<tr>
<td>0-115.5i</td>
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<td>-300-241.2i</td>
<td>-300-241.2i</td>
</tr>
<tr>
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<td>Desired Close Loop κ(Φ)</td>
<td>Achieved Cond. No. κ(Φ)</td>
</tr>
<tr>
<td>10.8555</td>
<td>1</td>
<td>2.5196</td>
</tr>
</tbody>
</table>

Table 1. Eigenvalues for the structural model of the morphing wing

V. Conclusions and Future Directions

The robust eigenstructure assignment methodology has been applied to redundantly actuated dynamical systems. For application to second order mechanical systems, a partial eigenstructure assignment has been incorporated. A case for the continuous time optimal control formulation for the allocation problem has been made. The merit of the optimal allocation methodology for application in different situations such as
control constrained problems needs further investigation and comparison with the existing methods of control allocation. It is felt that the methodology presented would be useful in the apriori calculation of preferred redundant inputs which could subsequently be used by an existing optimization method for allocating control efforts to individual actuators. For smart materials with a very high order of redundancy, an algorithm to aggregate the control input and make them into finite groups, an adaptive aggregation strategy is proposed based on the local controllability index. Efforts are on the way to demonstrate the proposed theoretical developments will be investigated on the morphing wing model in the wind tunnel.

Acknowledgments

The authors wish to acknowledge the support of the Texas Institute of Intelligent Bio-Nano Materials and Structures for Aerospace Vehicles, funded by NASA Cooperative Agreement No. NCC-1-02038. Any opinions, findings conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Aeronautics and Space Administration.

References

Figure 6. Conditioning of the control influence matrix with mode addition
Figure 7. Morphing Wing experiment