Automatic Differentiation Based Dynamic Model for a Mobile Stewart Platform

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Abstract

This paper presents a novel approach to dynamical modeling and uses this approach to model the dynamics of a Stewart platform. The Stewart platform is used for a ground-based next generation high-fidelity test bed for modeling and simulating the interactions of two or more interacting on-orbit satellites during various phases of proximity operations. Satellite motions will be emulated by mounting the satellite models on the Stewart platform. Comprehensive dynamic models will be required for modeling and controlling the motion of this system, so that the envisioned motion can actually approximate on-orbit behavior, even during proximity operations involving dynamic contact events. Automatic differentiation, which operates on kinematic descriptions of the Stewart platform, while invoking hidden software tools for numerically generating and assembling the equations of motion, is utilized for analyzing the dynamic response of the system. This approach helps free analysts from having to derive, code, and validate the sensitivity models and enables development of a generic code for multi-body systems. The results can be used to generate various simplified models and help understand how variations of the system parameters affect the dynamic response.

Introduction

We model and simulate the interactions of two or more interacting on-orbit satellites during various phases of proximity operations by considering a ground-based high-fidelity test bed having two bodies, each of which has six DOF (degrees of freedom). Satellite motions are emulated by using the satellite dynamic models to drive the motion of Stewart platforms. The Stewart platform is a six DOF multi-body system, in which the payload platform is connected to a base by six extensible legs. This system, when implemented with quality linear actuators, has many advantages such as high structural rigidity, high precision positioning capability, and high force-output-to-manipulator-weight ratio. It has a small to moderate workspace but can cover a wide range of frequency.
Since the Stewart platform was proposed by Stewart in 1965 [1], it has evolved into a very popular mechanism and has been the topic of several research efforts in mechanics and robotics. In 1995 alone, more than 50 papers appeared. Compared with the extensive research on kinematics analysis, research papers on dynamics are relatively rare. A detailed fully nonlinear dynamics derivation that accounts for all moving mass in the thirteen body system has not been carried out so far. Do and Yang use Newton-Euler method to solve the inverse dynamics of the platform assuming the legs are symmetrical and thin [2]. Liu et al [3] developed the equation of motion using Lagrangian approach under the assumption that each leg can be modeled by one moving point. Ji [4] studied the effects of leg inertias on the Stewart platform dynamics. Dasgupta and Mruthyunjaya [5] developed a complete formulation of the inverse dynamics problem through the Newton-Euler approach, which is shown to be well-suited for parallel computation, but the math model became very large and complex.

In this paper, Lagrange’s approach is adopted to formulate the equation of motion for the Stewart platform, but we solve the problem via a new path. Novel means of utilizing automatic differentiation are used to generate and solve the equations using only high level geometric and kinematic descriptions of the system. Automatic Differentiation (AD) provides the accurate and efficient evaluation of derivatives for functions defined by computer programs [6]. This method, while theoretically exact, does not require one to derive or program the analytical form for the equations of motion, and also does not have the limitations of the traditional numerical differentiation method. A recursive chain rule-based evaluation technique is used for building numerically evaluated, analytically exact partial derivative models with respect to user-defined sets of independent variables [7], this method can get the fully nonlinear dynamic response (solution of the exact differential equations) while requiring little user intervention to derive or program the equation of motion.

From 1980s to now, several tools for AD have been developed. The chain rule can be used in a forward or backward mode and the AD is implemented by operator overloading or source transformation. Operator overloading is a recent computing approach for relegating the differentiation process to be automatically performed without requiring any derivation or coding; source transformation requires sophisticated compiler-like software to read in a computer program, determine which statements require differentiation, and then via a semi-automated process, generate a new version of the original program augmented with statements to calculate derivatives [8]. The potential power of AD to save human calendar days in a broad class of applications is immediately obvious; the efficiency as regards storage and execution speed is a much more complicated story, however [9]. The OCEA AD environment used in this paper relies exclusively on operator overloading. A detailed mathematical description for OCEA can be found in [7] and [10]. Some applications of OCEA in mechanical systems can be found in [11]. Detailed description of this indirect approach for Lagrangian method can be found in [9]. The advantages of the velocity level Lagrangian approach with AD to formulate the equation of motion is captured in the following discussion.
Firstly, analysts are freed from having to derive, code, and validate the equation of motion when using AD. Once the formulation has been set up, a generic code has been developed. OCEA allows analysts to flexibly change their choice of general coordinates for the Lagrangian methods, and their control variables from work and task space to joint length space. Even under the condition that the geometry of the configuration has been changed, the only work that is needed is to revise the velocity level kinematics description of the system.

Secondly, we develop a dynamical model for the Stewart platform with a complete and general parameter, structure, mass, and inertial distribution. A fully nonlinear model for Stewart platform is important because most of the previous work done in this field has imposed significant simplification. Also we point out here, although such a huge amount of computation makes this method not very attractive for real-time modes, we can use this method to judge specific simplified models, investigate the general difference between the complete model and simplified models, and learn which parameters will have a great effect on system dynamic response. We will give one example in this paper.

Thirdly, this method considers the situation when the base body is moving in x-y plane with 3 DOF. The six DOF Stewart platform “rides” on a three DOF mobile robot which can provide large translation and rotation motions. Using OCEA, we just need to add kinematic equations for the base body and all the derivations can be carried out automatically.

The technical developments for the paper are organized as follows. Firstly, we briefly summarize the approach we use in this paper. We use a Lagrangian approach but from an indirect path. We build the velocity-level system kinematic description and “leave the rest of work to OCEA”. As pointed out in [9], such kinematic analysis is at the heart of dynamic system analysts because from these, we can form equation of motion by most available methods (e.g. D’Alembert, Kane, Newton/Euler, Lagrange, ...). Secondly, we describe the geometry of the Stewart platform and different co-ordinate frames we use to formulate the equations of motion. Then, we analyze a few simplified cases to explain the method clearly, and then look at the general case. After that, we use Lyapunov method to design a simple control law for the Stewart platform tracking problem. We compare the generalized forces that are needed for our complete model and one simplified model that was proposed in [12]. Finally, we present our conclusions.

**Velocity level kinematic analysis approach for Lagrangian equation**

The most general form of the Lagrangian equations of motion, including generalized forces and constraint forces is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \mathbf{Q} + \mathbf{C}^T \lambda$$  

(1)
subject to constraints in the Pfaffian form

\[ Cq = b \]  

where the Lagrangian is defined as \( L = T - V \), \( q \) is the generalized coordinates, \( \dot{q} \) is the generalized velocities, \( Q \) is the generalized forces, \( C(q) \) is the constraint matrix, and \( \lambda \) is the Lagrange multiplier vector. Suppose the model is an \( n \)-body system and the number of generalized coordinates is \( m \). The basic steps of the following derivatives are presented in [13]. Here we give the corresponding formulations that are needed for our AD approach. Introducing the following transformations:

\[ V_i = A_i(q)q \]  
\[ \omega_i = B_i(q)\dot{q} \]  

The kinetic energy \( T = \frac{1}{2} \sum_{i=1}^{n} (m_i V_i^T V_i + \omega_i^T I_i \omega_i) \) can be expressed as:

\[ T = \frac{1}{2} \dot{q}^T M(q)\dot{q} \]  

From Equation 5, we compute:

\[ \frac{\partial M}{\partial q} = \sum_{i=1}^{n} \left( m_i \frac{\partial A_i}{\partial q} A_i + m_i A_i^T \frac{\partial A_i}{\partial q} + \frac{\partial B_i}{\partial q} I_i B_i + B_i^T I_i \frac{\partial B_i}{\partial q} \right) \]  

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) = \frac{d}{dt} (M(q)\dot{q}) \]  
\[ = M(q)\ddot{q} + \left( \sum_{i=1}^{n} \frac{\partial M}{\partial q} q_j \right) \dot{q} \]  

\[ \frac{\partial T}{\partial q} = \frac{1}{2} \dot{q}^T \frac{\partial M}{\partial q} \dot{q} \]  

Then the Lagrangian equation can be rewritten as:

\[ M(q)\ddot{q} + \left( \sum_{i=1}^{n} \frac{\partial M}{\partial q} q_j \right) \dot{q} - \frac{1}{2} \dot{q}^T \frac{\partial M}{\partial q} \dot{q} - \frac{\partial V}{\partial q} = Q + C^T \lambda \]  

which is subject to constraint Equation 2. From Equations 3 and 4, it is easy to get:

\[ A_i = \frac{\partial R_i}{\partial q} = \frac{\partial \mathbf{R}_i}{\partial q} \]  
\[ B_i = \frac{\partial \omega_i}{\partial q} \]  

Now, to model a multi-body system, the work needed is to choose the generalized coordinates, solve \( A_i, B_i \) for each body, and combine Equations 2 and 9 together to solve for the numerical results of the motion. The partial derivatives that are needed in Equations 6, 7, 8, 9 are obtained from OCEA directly.
Geometry of Stewart platform and frames

In this paper, we focus on a Stewart platform configuration which is shown in Figure 1(a). The top body and the base body are connected together by six extensible legs. The legs are attached to the triangular top body and to the base body by revolute joins. To study the dynamics of the legs in detail, we decompose the legs to two parts: the fixed part is linked to the base body and the moving part is linked with the top body. The diagram of the legs is shown in Figure 1(b). The dimensions of the top body and the base body frames are shown in Figure 2(a) and Figure 2(b). Vectors locating the base body center of mass relative to the origin of the inertial frame is:

\[ \mathbf{R}_{\oplus} = [i_{\oplus} R_{b\oplus} + j_{\oplus} R_{b\oplus} + k_{\oplus}] \]

(12)

Here \( \{i, j, k\} \) are three orthogonal unit vectors defined in the inertial frame. The shorthand notation \( \mathbf{i} R_{b\oplus} \) is used to specify that the vector components are taken along the unit direction vectors of the “\( i \)” frame, which is inertial frame here. This allows us to use symbol \( \mathbf{i} R_{b\oplus} \) to denote the vector, as in Equation 12, as the matrix of i-frame...
components \( i^{\prime}R_{b\oplus} = [R_{b\oplus,x} R_{b\oplus,y} R_{b\oplus,z}]^T \), with the meaning clear in the context of a particular equation. It should be noted that in our approach, the base body is not fixed in the inertial frame. To do the on-ground spacecraft proximity emulation, we will suppose the base body can move in the x-y plane with large x, y and yaw motion. The coordinates of the 6 reference points in the base body are:

\[
R_{bi} = R_{b\oplus} + r_{bi}, \quad i = 1, 2, 3, 4, 5, 6
\]  
(13)

\([\hat{b}_{tx}, \hat{b}_{ty}, \hat{b}_{tz}]\) is the triad of unit vectors for the base body-fixed frame B. \( r_{bi} \) denotes the inertial relative position vectors from the center of mass to the 6 reference points \( B_i \). In the B(base) frame, these vectors have constant components:

\[
\begin{align*}
^b r_{b1} &= \frac{\sqrt{3}}{6} (2b + d) \hat{b}_{tx} + \frac{1}{2} d \hat{b}_{ty} \\
^b r_{b2} &= -\frac{\sqrt{3}}{6} (b - d) \hat{b}_{tx} + \frac{1}{2} (b + d) \hat{b}_{ty} \\
^b r_{b3} &= \frac{\sqrt{3}}{6} (b + 2d) \hat{b}_{tx} + \frac{1}{2} \hat{b}_{ty} \\
^b r_{b4} &= -\frac{\sqrt{3}}{6} (b + 2d) \hat{b}_{tx} + \frac{1}{2} \hat{b}_{ty} \\
^b r_{b5} &= -\frac{\sqrt{3}}{6} (b - d) \hat{b}_{tx} + \frac{1}{2} (b + d) \hat{b}_{ty} \\
^b r_{b6} &= \frac{\sqrt{3}}{6} (2b + d) \hat{b}_{tx} - \frac{1}{2} \hat{b}_{ty}
\end{align*}
\]  
(14a-19a)

where \( b \) and \( d \) are defined in the Figure 2(b) and the superscript \( B \) is used to specify that the components of the vector are taken in the base body frame. Vectors locating the top body center of mass relative to the origin of the inertial frame

\[
R_{t\oplus} = R_{t\oplus,x} \hat{i} + R_{t\oplus,y} \hat{j} + R_{t\oplus,z} \hat{k}
\]  
(20)

The coordinates of the 3 reference point in the top body is:

\[
R_{ti} = i^{\prime} R_{t_i} = R_{t\oplus} + r_{ti}, \quad i = 1, 2, 3
\]  
(21)

\([\hat{b}_{tx}, \hat{b}_{ty}, \hat{b}_{tz}]\) is the triad of unit vectors in the top body-fixed frame. \( r_{ti} \) denotes the inertial relative position vectors from the center of mass to the 3 reference points \( T_i \).

\[
\begin{align*}
^\prime r_{t1} &= \frac{\sqrt{3}}{6} a \hat{b}_{tx} + \frac{1}{2} a \hat{b}_{ty} \\
^\prime r_{t2} &= -\frac{\sqrt{3}}{3} a \hat{b}_{tx} \\
^\prime r_{t3} &= \frac{\sqrt{3}}{6} a \hat{b}_{tx} + \left(-\frac{1}{2} a \hat{b}_{ty}\right)
\end{align*}
\]  
(22a-24a)

\( a \) is the length of the assumed equilateral triangle as shown in Figure 2(a) and the components of the position vectors are taken in the top body frame.
Kinematic analysis for the base body

Let the direction cosine matrix $[BI]$ transforms vectors written in the base body frame into vectors expressed in the inertial frame. Considering the car is moving in the x-y plane, we use the following form to solve for the inertial position of the six reference points in the base body

$$ R_b(i) = R_{\oplus} + [BI]^B r_{bi} \quad (25) $$

$$ [IB] = [BI]^T = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \quad (26) $$

we assume $R_{\oplus}$ is some known functions of time or is alternatively the solution of additional differentiation equation of motion. $\phi$ is the yaw angle of the base body.

Kinematic analysis for the top body

In this and next section, we focus on the kinematics analysis of top body and legs and find $A_i$ and $B_i$ for each body. By doing so, we will develop all the terms that are needed for using Lagrangian method to formulate the equation of motion. We choose the mass center location of the top body $R_c = [X_c, Y_c, Z_c]^T$ and the three Euler angles $\theta_1, \theta_2, \theta_3$ of the top body as the generalized coordinates and $q = [X_c, Y_c, Z_c, \theta_1, \theta_2, \theta_3]^T$.

$$ A_{top} = \frac{\partial R_c}{\partial q} \quad (27) $$

and in this situation,

$$ A_{top} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (28) $$

$$ B_{top} = \frac{\partial \omega}{\partial q} \quad (29) $$

$B_{top}$ is constructed using the form in [13]. Let the direction cosine matrix $[TI]$ transforms vectors written in the top body frame into vectors expressed in the inertial frame. $[TI]$ can be expressed by $[\theta_1, \theta_2, \theta_3]$ using 1-2-3 order. $[TI] = [TI]^T$ can be found in [13]. Equation 21 is calculated explicitly using the following form to get the inertial position of the three reference points in the top body

$$ R_{ti} = R_{\oplus} + [TI]^t r_{ti}, \quad i = 1, 2, 3 \quad (30) $$
Kinematic analysis for the legs

We choose the plane whose three vertex are $T_1$, $B_1$ and $B_2$ as an example to show the process to resolve for $A_i$ and $B_i$ for the legs. The geometry is shown in the Figure 1(b). The unit vector pointing from leg 2 to leg 1 is $\hat{r}_{21}$, and the unit vector pointing from leg 4 to leg 3 is $\hat{r}_{43}$. The normal vector to this plane is $\hat{r}_{\text{normal}}$ and

$$\hat{r}_{\text{normal}} = \hat{r}_{21} \times \hat{r}_{43}$$

(31)

The third unit vector for the body frame of leg 1 and leg 2 is $\hat{r}_{21}^{\perp}$, which is computed from the following form:

$$\hat{r}_{21}^{\perp} = \hat{r}_{\text{normal}} \times \hat{r}_{21}$$

(32)

Let $[L_1 I]$ transforms vectors written in the leg 1 frame into vectors expressed in inertial frame. The direction cosine matrix $[L_1 I]^T = [I L_1] = [r_{21}^T, r_{212}^T, r_{\text{normal}}^T]$. For leg 1, its length is denoted as $|L_1|$ and the location of its center of mass can be calculated by:

$$\vec{r}_{c_{\text{leg1}}} = R_{\text{b}}(1) - [L_1 I]|L_1|/2, 0, 0]^T$$

(33)

and we can solve for all the terms that we need using the following equations:

$$A_{\text{leg1}} = \frac{\partial \vec{r}_{c_{\text{leg1}}}}{\partial \vec{q}}$$

(34)

$$[\omega^*] = -\left(\frac{d}{dt} [L_1 I] [L_1 I]^T \right)$$

(35)

$$[\omega^*]^T = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} = [L_1 I] \chi \dot{\vec{q}}$$

(36)

$$\chi = \begin{bmatrix} \frac{\partial |L_1 I|^T}{\partial \vec{q}_1} & \frac{\partial |L_1 I|^T}{\partial \vec{q}_2} & \frac{\partial |L_1 I|^T}{\partial \vec{q}_3} & \frac{\partial |L_1 I|^T}{\partial \vec{q}_4} & \frac{\partial |L_1 I|^T}{\partial \vec{q}_5} \end{bmatrix}$$

(37)

The coefficient of $\dot{\vec{q}}$ is computed in the background by OCEA. Taking the corresponding elements from Equation 36, we can solve for $B_{\text{leg1}}$ as:

$$B_{\text{leg1}} = \frac{\partial \vec{w}}{\partial \vec{q}} = \begin{bmatrix} \frac{\partial \omega_1}{\partial \vec{q}} & \frac{\partial \omega_2}{\partial \vec{q}} & \frac{\partial \omega_3}{\partial \vec{q}} \end{bmatrix}^T$$

(38)

For leg 2, which is the upper leg of leg 1, the location of its center of mass is:

$$\vec{r}_{c_{\text{leg2}}} = R_{\text{b}}(1) + [L_2 I]|L_2|/2, 0, 0]^T$$

(39)

$[L_2 I]$ denotes the direction cosine matrix between the inertial frame and leg 2 frame and $|L_2|$ is the length of leg 2. As before, $A_{\text{leg2}}$ can be obtained from OCEA automatically. In addition, because leg 2 will only have translational relative motion to leg 1, so $B_{\text{leg2}}$ is the same as $B_{\text{leg1}}$. The above procedures can be applied for the other 10 legs to get the individual items of $A_i$ and $B_i$. After obtaining $A_i$ and $B_i$ for each body, we assemble them in Equation 5, and the Lagrangian Equation 9 can be constructed easily.
Simulation results

In this section, we design a control law for the Stewart platform tracking problem and compare the generalized forces that are needed for our complete model and one simplified model used in [12]. In this simplified model, the leg dynamic parts have been neglected. Having some initial state $q_{T_0}$ and final desired condition $q_{T_f}$, we construct a “for example” target motion by connecting a 3rd order spline curve between the initial and final states.

$$q_x(t) = q_{T_0} + f(t)(q_{T_f} - q_{T_0}) \quad (40)$$

$$f(t) = \tau^2(3 - 2\tau) \quad (41)$$

$$\tau = t/T_f \quad (42)$$

Notice that this gives the simplest “rest-to-rest” motion of the system. The Lyapunov method is used to design a feedback control law and the Lyapunov function is defined as a quadratic measure of tracking errors:

$$U = \frac{1}{2}(q - q_r)^TK_1(q - q_r) + \frac{1}{2}(\dot{q} - \dot{q}_r)^TK_2(\dot{q} - \dot{q}_r)$$

$$= \frac{1}{2}\delta q^TK_1\delta q + \frac{1}{2}\dot{\delta q}^TK_2\dot{\delta q} \quad (43)$$

where $\delta q = q - q_r$, $\dot{\delta q} = \dot{q} - \dot{q}_r$. In order for Equation 43 to qualify for a Lyapunov function, we require $K_1$ and $K_2$ to be positive definite. Taking the time derivative of $U$, we find:

$$\dot{U} = \delta \dot{q}^T(K_1\delta q + K_2\dot{\delta q}) \quad (44)$$

where $\delta \dot{q} = \ddot{q} - \ddot{q}_r$. We can set $K_1\delta q + K_2\dot{\delta q} = -K_3\delta q$ and $K_3 > 0$ to make sure $\dot{U} < 0$. The ideal close-loop error dynamics is:

$$\ddot{\delta q} + C_1\dot{\delta q} + C_2\delta q = 0 \quad (45)$$

Here $C_1$ and $C_2$ are free positive feedback gains we can choose. Depending on the desired response for the ideal error dynamics, we can pick up these two matrixes by using Eigenstructure assignment regulator [14]. Since we imposing Equation 45, we can eliminate $\dot{\delta q}$ in $\ddot{\delta q} = \ddot{q} - \ddot{q}_r$, and substitute into the equation of motion (Equation 9); this permits to solve for the actuator feedback law by inverse dynamics. Choosing the frequencies and damping factors for the ideal error dynamics as 1 rad/s and 0.7, we get a satisfactory tracking. Figure 3 shows the desired path for the platform. Figure 4 shows the motion of the base body and the top body (this is used to represent the motion of the platform), when the platform has some initial errors. Figure 5 shows that the tracking error of the platform decreases with time. However, such ideal dynamics is not exact in applications and the resulting feedback linearized input-output behavior may be a poor approximation of the actual system’s behavior [15]. To cope with the presence of bounded model errors and disturbances, an adaptive feedback control law will be designed in the later work. Relative errors of the
generalized force that are needed for the complete model and the simplified model are computed by the following way:

$$\tau = \frac{Q_c - Q_s}{Q_c}$$ \hspace{1cm} (46)

$Q_c$ is the generalized force for the complete model and $Q_s$ is the force for the simplified model. In simulation I, when the top body weights 307.6kg, the top leg weights 5.67kg and the bottom leg weights 22.7kg, at most of the time, the relative errors are less than 10%, as shown in Figure 6. This means that the simplified model can likely be the basis for an adaptive feedback law designed to accommodate for model er-
errors and disturbances of this magnitude. In simulation II, when the top body weights 153.8kg, while the top leg weights 22.7kg and the bottom leg weights 12.8kg, we find that frequently, the relative errors are between 20\% and 30\%, which means using the simplified model will cause unacceptable errors unless we improve the model or use an adaptive controller.
Conclusions

In this paper, we present a velocity level kinematics description approach for dynamic modeling appropriate for invoking the OCEA automatic differentiation method to generate and solve the equations of motion. This approach generates a generic code for a variety of multi-body systems and can save substantial human efforts to derive and code lengthy expressions yet preserve the ability to invoke model simplifications in a flexible fashion.

The proposed approach has been utilized for a complete dynamic modeling for a Stewart platform. A Lyapunov method is used to design a tracking control law when the base body is moving and rotating in the x-y plane. The complete model is compared with one simplified model to show another advantage of our approach of being able to help the analyst learn the system parameters’ effects on dynamic response simulations and enable confident control law design studies.

References


