A Homotopic Galerkin Approach to the Solution of the Fokker-Planck-Kolmogorov Equation

Suman Chakravorty

***Abstract*— In this paper, we present a homotopic Galerkin approach to the solution of the Fokker-Planck-Kolmogorov equation. We argue that the ideal Hilbert space to approximate the exact solution, ψ^*, is the space $L_2(d\Psi^*)$ where $d\Psi^*$ is the probability measure induced on $\Re^n$ by the solution ψ^* itself. We show that given an initial approximation of the exact solution which is sufficiently close to the exact solution, we can obtain the exact solution iteratively by propagating the solution over sufficiently small time intervals using the Galerkin projection method. Further, we show that given a family of dynamical systems, $\mathcal{D}_p$, indexed by the homotopy parameter $p \in [0, 1]$, where the dynamical system corresponding to $p = 0$ is a system whose associated Fokker-Planck equation can be solved and $p = 1$ is the dynamical system of interest, and such that the associated solutions to the corresponding Fokker-Planck equations can be changed smoothly by slowly varying the homotopy parameter, the exact solution can be obtained recursively, using the Galerkin projection method, starting with the solution to the Fokker-Planck equation associated with the dynamical system $\mathcal{D}_0$.**

I. Introduction

In this paper, we concern ourselves with approximate solutions to the Fokker-Planck equation. Consider the stochastic dynamical system governed by the Ito differential equation

$$dx = f(x)dt + g(x)dW, \tag{1}$$

where W Wiener process with intensity Q. Its well known that the probability density function (pdf) of the state x of the dynamical system obeys the Fokker-Planck Kolmogorov equation [1]:

$$\frac{\partial p}{\partial t} = -\sum_i \frac{\partial f_i p}{\partial x_i} + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 (gQg^t)_{ij} p}{\partial x_i \partial x_j}. \tag{2}$$

The Fokker-Planck equation is a linear second order partial differential equation of parabolic type. Analytical solutions to the Fokker-Planck equations are exceedingly rare and usually numerical methods have to be employed in order to find an approximate solution. The numerical solution of the Fokker-Planck equation is difficult due to the following reasons:

- **Positivity and normality:** The approximate solution to the Fokker-Planck equation resulting from any numerical method has to be positive and its integral over its domain needs to be unity, since the solution has to satisfy the necessary conditions for being a pdf.
- **Domain:** The domain of the pdf is infinite and this implies that any numerical method has to assume a large enough domain, such that the support of the pdf can be assumed to be contained within it for all practical purposes.

Suman Chakravorty is with the faculty of Aerospace Engineering, Texas A&M University, College Station, schakrav@aero.tamu.edu

In this paper, we shall consider the latter of the above mentioned problems in the solution to the Fokker-Planck equation. Various different methods have been proposed for the approximate solution of the Fokker-Planck equation [1], [2]: iterative solution based on an associated integral equation [3], [4], solutions based on an eigenfunction expansion of the pdf [5], [6], solutions based on the Galerkin residual projection method [7], [1], solutions based on perturbation methods [8], equivalent linearization methods [9], [10] and Gaussian closure techniques [11]. Please see references [1], [2] for a detailed exposition on each of the above mentioned methods. In this paper, we shall concern ourselves with the Galerkin residual projection method for the solution of the Fokker-Planck equation.

The Galerkin method is an elegant and powerful method for solving partial differential equations [12], [13]. The Galerkin method has been extensively used in the solution to the Fokker-Planck equation [7], [14], [15], [16], [17], [18], [19] and is also sometimes known as the "weighted residual method" in the literature [19]. The "traditional" Galerkin projection method projects the residual (or equation error) onto the basis function themelves in order to solve for the coefficients. In the weighted residual method, as in the above references, the integrals involved in the projection equation are weighted by a suitable function. Frequently, this is the pdf obtained for the system using Gaussian Closure methods or statistical linearization methods [17], [18], [19]. It is asserted in all these papers that the choice of the weighting function is very critical to the accuracy of the approximation. In this paper, we argue that the best weighting function that should be used in order to evaluate the projection equation is the exact pdf itself, i.e., the exact solution to the Fokker Planck equation. This corresponds to approximating the solution in the space $L_2(d\Psi^*)$ where $d\Psi^*$ is the probability measure induced on $\Re^n$ by the exact solution to the Fokker-Planck equation, ψ^*, as opposed to projecting the solution in the standard L_2 space (i.e., the L_2 space equipped with the Lebesgue measure on $\Re^n$). However, the drawback of this method is that we do not know the exact pdf since we are in fact solving for it. In this paper, we show that if we have a sufficiently close approximation to the actual pdf, then the exact solution can be obtained for the time interval of interest recursively, by propagating the solution over sufficiently

small sub-intervals of time (a time homotopy). Moreover, we also show that given a homotopy of dynamical systems $\mathcal{D}_p, p \in [0,1]$, where $\mathcal{D}_1$ is the system of interest and $\mathcal{D}_0$ is a system whose response is known (the associated Fokker-Planck equation has a known solution), i.e., the solution to the Fokker-Planck equation corresponding to $\mathcal{D}_0$ can be changed in a smooth fashion to obtain the solution of the Fokker-Planck equation corresponding to $\mathcal{D}_1$ by slowly varying the homotopy parameter p; then starting with the solution of the system $\mathcal{D}_0$, we can obtain the solution of the Fokker-Planck equation corresponding to the dynamical system of interest, $\mathcal{D}_1$, in a recursive fashion. Moreover, we argue that the methodology is inherently computationally tractable as it corresponds to solving a sequence of smaller systems of differential equations, as opposed to one large system of ODEs, as in the case of the traditional Galerkin approach to the solution of the Fokker-Planck equation.

The rest of the paper is organized as follows. In section 2, we present the Fokker-Planck equation and the Galerkin method to solve it. In section 3, we present a modified problem formulation in order to obtain a solution to the Fokker-Planck equation using the Galerkin projection method. We also discuss the interpretation of the Galerkin projection method in terms of approximations in a suitable Hilbert space. In section 4, we present the homotopic Galerkin method for solving the problem posed in section 3. In section 5, we draw conclusions on the work presented in this paper.

II. The Fokker-Planck Equation

Consider the stochastic dynamical system governed by the Ito differential equation

$$dx = f(x)dt + g(x)dW, \tag{3}$$

where x is a vector valued random variable and W is a Weiner process with intensity Q. It is known that the probability density function of the state x, $p(x)$, satisfies the Fokker-Planck-Kolmogorov equations:

$$\frac{\partial p}{\partial t} = -\sum_i \frac{\partial f_i p}{\partial x_i} + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 (gQg^t)_{ij} p}{\partial x_i \partial x_j}. \tag{4}$$

As mentioned before the soltuion of the Fokker-Planck equation is made difficult due to the problems of "positivity", "normality" and "domain" of the desired solution. In this paper, we shall concentrate on the last item of the above mentioned issues.

The Galerkin projection method is a general method to solve partial differential equations and is often used to solve the Fokker-Planck equations. In the following, we shall present a brief overview of the Galerkin Projection method, with application to the solution of the Fokker-Planck equations. Please refer to [12], [13] for further details about this elegant and powerful method to solve PDEs. Let $p(t,x)$ represent the time varying solution to the Fokker-Planck equation (4). Let $\{\phi_1(x), \cdots, \phi_n(x), \cdots\}$ represent a set of basis vectors defined on the domain D and assume that $p(t,x)$ is restricted to domain D for all time t. The Galerkin method makes the assumption that the pdf $p(t,x)$ can be written as a linear combination of the basis vectors ϕ_i with time varying coefficients, i.e.,

$$p(t,x) = \sum_{i=1}^{\infty} c_i(t)\phi_i(x). \tag{5}$$

The key to obtaining the solution to the pdf $p(t,x)$ is to find the coefficients $c_i(t)$ in the above equation. The Galerkin method achieves this by truncating the expansion of $p(t,x)$ for some finite N and projecting the equation error onto the set of basis vectors $\{\phi_i\}$, i.e.,

$$p(t,x) \approx \sum_{i=1}^{N} c_i(t)\phi_i(x), \tag{6}$$

and

$$< \frac{\partial p}{\partial t} - L_{FP}(p), \phi_j(x) >= 0, \tag{7}$$

where $< .,. >$ represents the inner product on $L_2(D)$ and L_{FP} represents the linear differential operator corresponding to the Fokker-Planck equation. Note that the above represents a finite set of differential equations, which can be solved for the coefficients $c_i(t)$, given by:

$$\sum_{i=1}^{N} \dot{c}_i < \phi_i, \phi_j > - \sum_{i=1}^{N} c_i < L_{FP}(\phi_i), \phi_j >= 0, \forall j. \tag{8}$$

In the next section, we shall discuss the implications for the Galerkin projection method with respect to the issue of the domain of the solution to the Fokker-Planck equation.

III. Motivation and Problem Formulation

In this section, we formulate the problem of approximating the solution of the Fokker-Planck equation as one of finding the best approximation of a vector in a given subspace of a suitably defined Hilbert space. First, we shall explore the connection between the Hilbert projection theorem and the Galerkin projection method.

Recall the Galerkin projection equations for solution of the Fokker-Planck equations:

$$\sum_{i=1}^{N} \dot{c}_i < \phi_i, \phi_j > - \sum_{i=1}^{N} c_i < L_{FP}(\phi_i), \phi_j >= 0, \forall j. \tag{9}$$

Generally, the solution to the above equation assumes that we have a sufficiently large domain D such that the support of the solution $p(t,x)$ can be assumed to be contained in it for all practical purposes. However, though it is always possible to do this, if the domain D becomes too large, then the number of basis elements required to approximate the pdf becomes large and consequently, the resulting system of differential equations becomes computationally intractable. The support of the solution, $p(t,x)$, to the Fokker-Planck equations would in general be time varying, and unknown a priori. At this point, we digress to discuss the interpretation of the Galerkin projection equations in terms of the Hilbert projection theorem [20].

Consider $L_2(D)$, the standard L_2 space equipped with the

Lebesgue measure on the domain D. Let $\psi^*(t,x)$ denote the exact solution to the Fokker-Planck equation (4) for any time t. Suppose that we seek an approximation to the solution $\psi^*(t,x)$ in the span of $B = \{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\}$ for all time t. Then, by the Hilbert projection theorem [20], the optimal solution is given by the vector $\hat{\psi}(t) = \sum_{i=1}^{N} c_i(t)\phi_i(x)$, where the time varying coefficients can be found solving the following normal equations:

$$\sum_i c_i(t) < \phi_i, \phi_j > = < \psi^*, \phi_i >, \forall j, \forall t \in [0,T]. \quad (10)$$

It follows that the coefficients $c_i(t)$ obey the following differential equation (obtained by differentiating the above equation):

$$\sum_i \dot{c}_i(t) < \phi_i, \phi_j > = < \frac{\partial \psi^*}{\partial t}, \phi_j >, \forall j, \forall t \in [0,T], \quad (11)$$

with the initial condition $c_i(0)$ such that $\sum_i c_i(0)\phi_i(x)$ is the best approximation of $\psi^*(0)$ in the subspace B. Note that the solution $\hat{\psi}$ minimizes the L_2 norm of the error $(\hat{\psi} - \psi^*)$. However, the Hilbert projection approximation to the actual solution ψ^* cannot be obtained since ψ^* is unknown a priori. The Galerkin projection method projects the equation error, or the residual, onto the basis vectors since the actual solution has zero residual and thus, trivially satisfies the projection equations. The Galerkin projection equations are given by eqn. (9) above. The accuracy of the Galerkin method can be analyzed by studying the "closeness" of the Galerkin equations to the Hilbert projection equations, i.e., comparing the systems of equations (9) to (11). For the sake of simplicity let us assume that the basis vectors are orthonormal so that the Hilbert projection equations are modified to

$$\dot{c}_j(t) = < \frac{\partial \psi^*}{\partial t}, \phi_j >, \forall j. \quad (12)$$

Similarly, the Galerkin projection equations are modified to

$$\dot{c}'_j(t) = \sum_i c'_i < L(\phi_i), \phi_j > . \quad (13)$$

Note that if ψ^* can be approximated exactly in the subspace generated by $\{\phi_1, \cdots, \phi_n\}$, then the Galerkin and Hilbert projection equations are identical differential equations. The Galerkin projection equations can be viewed as a perturbation of the Hilbert projection equations and it can be shown that the difference between the solutions satisfies the following bound:

$$||\bar{c} - \bar{c}'|| \leq \epsilon(e^{Mt} - 1), \quad (14)$$

where $M = k||L||$, $||L||$ is the operator norm induced on the Fokker-Planck operator by the L_2 norm and k is a finite constant.

Hence, from the above development we can conclude that the Galerkin projection method can be made arbitrarily close to the Hilbert projection solution $\hat{\psi}$ by increasing the size of the basis set, since that would reduce the magnitude of ϵ. Also, note that given a basis set, we can approximate the Hilbert projection of the exact solution, $\hat{\psi}$, to any arbitrary accuracy for sufficiently small time. To summarize, the Galerkin method can be thought of as a method to obtain the best approximation to the solution of the Fokker-Planck equation in a given subspace of a suitably defined Hilbert space.

In the preceding development, the Hilbert space of interest was $L_2(D)$, where $D \subset \Re^n$, which was equipped with the Lebesgue measure. Suppose now that we want to approximate the solution of the Fokker-Planck equation ψ^* in the space $L_2(d\Psi^*)$, where $d\Psi^*$ is the probability measure induced on $\Re^n$ by the pdf ψ^*, i.e., we pose the following problem in order to approximate the solution to the Fokker-Planck equation (4):

P 1: Consider the stochastic dynamical system (3) and the Fokker-Planck equation (4) corresponding to it. Let $[0,T]$ be the time interval of interest and let $\psi^*(t,x)$ denote the exact solution of the given Fokker-Planck equation on $[0,T]$. Given a subset of basis vectors $B = \{\phi_1, \cdots, \phi_n\}$, find the best approximation to the solution ψ^* in the subspace spanned by B, which minimizes the weighted norm of the error e, defined by

$$|||e(t)||| = | << e(t), e(t) >> |^{1/2} = | \int_{\Re^n} |e(t,x)|^2 \psi^*(t,x) dx|^{1/2}, \quad (15)$$

for all time $t \in [0,T]$.

The following remarks are in order here.

- The space $L_2(d\Psi^*)$ constitutes the natural co-ordinates to evaluate any approximation to the exact solution ψ^*. Suppose δ is the error between the exact solution ψ^* and its approximation $\hat{\psi}$, then

$$|||\delta||| = | \int_{\Re^n} |\delta(x)|^2 \psi^*(x) dx|^{1/2}. \quad (16)$$

  Thus, loosely speaking, the pdf ψ^* implicitly defines the weighting to be associated with the error function as opposed to the Lebesgue measure which corresponds to applying a uniform weight over the whole domain. Hence, intuitively it is appealing to minimize the "naturally" weighted norm of the error function as opposed to any other kind of norm.
- The above problem definition makes an approximate solution to the Fokker Planck equation computationally tractable. Since the pdf $\psi^*(t,x)$ implicitly defines the domain of the problem, the number of basis functions required to approximate the solution would be much much smaller than if a large domain is used as in the "traditional" Galerkin approach. This is especially transparent for the case where we have basis functions with localized supports, i.e., a finite element type approach.

In the next section, we shall propose a method of solution to the above problem, namely the homotopic-Galerkin solution. We shall also analyze the performance of the methodology and prove its convergence to the best approximation in the sense of Problem $\mathcal{P}1$.

IV. A Homotopic Galerkin Solution to the Fokker-Planck Equation

We mentioned in the previous sections that the solution of the Fokker-Planck equations is rendered computationally difficult because of lack of information about the domain of the solution. In the previous section, we formulated problem $\mathcal{P}$ 1 which implicitly defines the domain of the differential equations in terms of the exact solution to the Fokker-Planck equations. In the sequel, we shall present a methodology to solve the above mentioned problem.

Let $\psi^*(t,x)$ represent the exact solution to the Fokker-Planck equation (4) corresponding to the stochastic dynamical system (3). Let $B = \{\phi_1, \cdots, \phi_N\}$ be a set of basis functions defined on $\Re^N$. We make the following assumption about the basis functions and the exact solution $\psi^*(t,x)$.

$\mathcal{A}$ 4.1: We assume that the function ψ^* can be approximated exactly in the subspace generated by B, for all time $t \in [0,t]$, i.e., there exist $c_1(t), \cdots, c_N(t)$, such that

$$\psi^*(t,x) = \sum_{k=1}^{N} c_k(t)\phi_k(x), \forall\, t \in [0,T], x \in \Re^n. \quad (17)$$

A. Time Homotopy

In the following, we shall develop the time homotopy which allows us to find a solution to problem $\mathcal{P}1$, given that we have a sufficiently good estimate of the exact solution ψ^*.

Let $\psi(t,x)$ be an approximation of $\psi^*(t,x)$ such that

$$||\psi - \psi^*|| \leq \epsilon, \quad (18)$$

for all $t \in [0,T]$. Recall problem $\mathcal{P}$ 1. We need to find an approximation to ψ^* in the span of B, $\hat{\psi}$, such that the error $|||\psi^* - \hat{\psi}|||$ is minimized for all time $t \in [0,T]$, where $|||.|||$ represents the L_2 norm induced on $\Re^n$ by the pdf ψ^*.

Due to the Hilbert projection theorem [20], the exact solution to Problem $\mathcal{P}$ 1 is given by the vector $\hat{\psi}$ such that $(\hat{\psi} - \psi^*)$ is orthogonal to B, i.e., $\hat{\psi} = \sum_k c_k\phi_k$ such that

$$\sum_k c_k < \phi_k, \phi_j > - < \psi^*, \phi_j >= 0, \forall j, \quad (19)$$

where $< .,. >$ represents the inner product on $L_2(\Re^n)$ induced by the probability measure corresponding to the pdf ψ^*. Differentiating the above equation, we obtain

$$\sum_k \dot{c}_k < \phi_k, \phi_j > + \sum_k c_k < \phi_k, \dot{\phi}_j > = 2 < \frac{\partial \psi^*}{\partial t}, \phi_j >, \forall j. \quad (20)$$

Note that in the above equation, ψ^* is required in order to evaluate the coefficients of the differential equation. However, we only have an approximation ψ of ψ^*. Thus, we shall use the Galerkin projection method to approximate ψ^* using ψ to define the required inner products. Let the inner product induced on $L_2(\Re^n)$ by the pdf ψ be represented by $<< .,. >>$, i.e.,

$$<< \phi_i, \phi_j >>= \int_{\Re^n} \phi_i(x)\phi_j(x)\psi(t,x)dx. \quad (21)$$

Then, following a development similar to the one shown in section 2, the Galerkin projection equations can be derived as the following:

$$\sum_k \dot{c}'_k << \phi_k, \phi_j >> - \sum_k c'_k << L(\phi_k), \phi_j >>= 0, \forall j. \quad (22)$$

We can view equation (22) as a perturbation of equation (20) and derive results for the closeness of the solutions of the two systems of equations, under certain assumptions. The equations can be written in the vector-matrix form as:

$$A(t)\dot{\bar{C}} + B(t)\bar{C} = F(t), \quad (23)$$

$$(A(t) + \Delta_3(t))\dot{\bar{C}}' + B(t)\bar{C}' = F(t) + \Delta_1(t) + \Delta_2(t), \quad (24)$$

where the first equation is the Hilbert projection equation and the second equation is the Galerkin projection equation written as a perturbation of the Hilbert equation. In the above equations

$$A(t) = [< \phi_k, \phi_j >], \quad (25)$$

$$B(t) = [< \phi_k, \dot{\phi}_j >], \quad (26)$$

$$F(t) = [f_j(t)],\ f_j(t) = 2 < \frac{\partial \psi^*}{\partial t}, \phi_j >, \quad (27)$$

$$\Delta_1(t) = [\delta_j^1(t)], \quad (28)$$

$$\delta_j^1(t) = \sum_k c'_k << L(\phi_k), \phi_j >> - < \frac{\partial \psi^*}{\partial t}, \phi_j >, \quad (29)$$

$$\Delta_2(t) = [\delta_j^2(t)], \quad (30)$$

$$\delta_j^2(t) = \sum_k c_k < \phi_k, \dot{\phi}_j > - < \frac{\partial \psi^*}{\partial t}, \phi_j > \quad (31)$$

$$\Delta_3(t) = [<< \phi_k, \phi_j >> - < \phi_k, \phi_j >]. \quad (32)$$

We have the following Lemma regarding bounds on the above quantities. Due to paucity of space, we omit the proofs in this paper.

Lemma 4.1: Let $||\psi - \psi^*|| \leq \epsilon$, and assumption $\mathcal{A}$ 4.1 hold, then the following bounds hold uniformly for all $t \in [0,t]$:

$$||\Delta_1(t)|| \leq K_1||L||\{||\bar{C}(t) - \bar{C}'(t)|| + \epsilon\},$$
$$||\Delta_2(t)|| = 0,$$
$$||\Delta_3(t)|| \leq K_3\epsilon,$$

where $||\Delta_1(t)||, ||\Delta_2(t)||$ represent the Euclidean norm of $\Delta_1(t)$ and $\Delta_2(t)$ respectively, $||\Delta_3(t)||$ represents the matrix norm of $\Delta_3(t)$ induced by the Euclidean norm on $\Re^n$ and K_3 is a finite constant.

We further make the following assumptions:

$\mathcal{A}$ 4.2: We assume that ϵ is small enough such that $||A^{-1}(t)||||\Delta_3(t)|| \leq 1$ uniformly for all $t \in [0,T]$.

$\mathcal{A}$ 4.3: We assume that $||A^{-1}(t)|| < K < \infty, \forall\, t \in [0,T]$. Moreover, we assume that $||L|| = L < \infty$.

With the above assumptions, we have the following lemma:

Lemma 4.2: Under Assumptions A 4.1, 4.2 and 4.3, and for sufficiently small ϵ, the following bound holds on the error between the solution to the Hilbert projection equation

20, $\bar{C}$, and the solution to the Galerkin projection equation 22, $\bar{C}'$,

$$||\bar{C}(t) - \bar{C}'(t)|| \leq (e^{C_2 t} - 1)\frac{C_1 \epsilon}{C_2}, \tag{33}$$

where C_1 and C_2 are finite constants, and $||.||$ represents the Euclidean norm on $\Re^n$.

Using the Lemma above and the last observation above, we can prove the following result.

Proposition 4.1: Given ψ_0, s.t. $||\psi_0 - \psi^*|| \leq \epsilon$ and that ϵ is sufficiently small, a sequence of functions $\{\psi_n\}_{n=0}^{\infty}$ can be constructed recursively, starting with ψ_0 such that $||\psi_n - \psi^*|| \to 0$ as $n \to \infty$.

Proof: Due to Lemma 4.2, if we use ψ_0 to approximate ψ^* in Problem $\mathcal{P}$ 1, we have:

$$||\bar{C}_0(t) - \bar{C}(t)|| \leq (e^{C_2 t} - 1)\frac{C_1 \epsilon}{C_2}, \forall t \in [0, t], \tag{34}$$

where $\bar{C}_0$ is the solution to Galerkin projection equation (22), and $\bar{C}$ is the solution to the Hilbert projection equation (20). Note that due to assumption $\mathcal{A}$ 4.1, the exact solution, ψ^*, is identical to the solution of the Hilbert projection equation.

Note that if $t \leq \frac{1}{C_2} log(1 + \frac{C_2}{C_1})$, then $(e^{C_2 t} - 1)\frac{C_1}{C_2} \leq 1$. Now, we have two possible cases:

(a) $T \leq \frac{1}{C_2} log(1 + \frac{C_2}{C_1})$

Then, we have that $||\bar{C}_0(t) - \bar{C}(t)|| \leq \beta\epsilon$, where $\beta < 1$ for all $t \in [0, T]$. Let $\psi_1 = \sum_k c_k^{(0)} \phi_k$, where $\bar{C}_0 = [c_1^{(0)}, \cdots, c_N^{(0)}]'$. Thus $||\psi_1 - \psi^*|| \leq \beta\epsilon$ due to assumption $\mathcal{A}$ 4.1. Hence if we use ψ_1 to approximate ψ^* in problem $\mathcal{P}$ 1, i.e., use ψ_1 in the Galerkin projection equation (22), we obtain that $||\bar{C}_1 - \bar{C}|| \leq \beta^2\epsilon$, where $\bar{C}_1$ is the solution to the Galerkin projection equation (22). By repeating this procedure, we see that $||\psi_n - \psi^*|| \to 0$.

(b) $T > \frac{1}{C_2} log(1 + \frac{C_2}{C_1})$

Divide $[0, T]$ into disjoint intervals $\bigcup_{k=0}^{n} [T_k, T_{k-1}]$ such that $T_0 = 0$, $T_n = T$ and $|T_k - T_{k-1}| \leq \frac{1}{C_2} log(1 + \frac{C_2}{C_1})$.

(i) Consider $[0, T_1]$. If we follow the procedure outlined in case (a) starting with ψ_0 on $[0, T_1]$, we have in the limit that $||\psi_n^{(1)} - \psi^*|| \to 0$ for all $t \in [0, T_1]$.

(ii) Consider the converged value for $\psi^*(T_1)$ from the answer in (i) above. Using this value for $\psi^*(T_1)$ and ψ_0 on $[T_1, T_2]$, we repeat the procedure outlined in (a) over the interval $[T_1, T_2]$. Then, it follows that we can obtain a sequence of functions $\{\psi_n^{(2)}\}$ s.t. $||\psi_n^{(2)} - \psi^*|| \to 0$ for all $t \in [0, T_2]$.

The above procedure can be repeated till we have the required solution over the whole of the interval $[0, T]$. Hence, we can obtain a sequence of functions $\{\psi_n\}$ recursively, starting with ψ_0, such that $||\psi_n - \psi^*|| \to 0$ as $n \to \infty$.

This completes the proof of the proposition.

Q.E.D.

B. Space Homotopy

In the previous section, we showed that if the initial guess ψ_0 is close enough to the exact solution ψ^* of Problem $\mathcal{P}$ 1, then through successive approximations and a time homotopy, the exact solution ψ^* can be obtained recursively starting with ψ_0. In this section, we answer the question, "what if ψ_0 is not close enough to ψ^*?". We use a space homotopy, in other words a one parameter family of stochastic dynamical systems indexed by the parameter $p \in [0, 1]$, i.e.,

$$\mathcal{D}_p : dx = f(x, p)dt + g(x, p)Q_p^{1/2} dW,\ p \in [0, 1], \tag{35}$$

where $\mathcal{D}_1$ corresponds to the dynamical system of interest and $\mathcal{D}_0$ corresponds to a stochastic dynamical system whose response is known, i.e., the Fokker Planck equation associated with it can be solved. Let $\psi_p(t, x)$ denote the solution of the Fokker Planck equation associated with dynamical system $\mathcal{D}_p$. We make the following assumption about the family of dynamical systems $\mathcal{D}_p$ and the solution of the associated Fokker Planck equations, ψ_p.

$\mathcal{A}$ 4.4: We assume that given any $p \in [0, 1]$, and any $\epsilon > 0$, there exists $\delta > 0$ s.t. for all $p' \in B_\delta(p)$, $||\psi_p(t, x) - \psi_{p'}(t, x)|| \leq \epsilon$, for all $t \in [0, T]$.

With the above assumption,we have the following result:

Proposition 4.2: Let ϵ_p be sufficiently small such that Proposition 4.1 is satisfied for any ψ satisfying $||\psi - \psi_p|| \leq \epsilon_p$. Let $\inf_{p \in [0,1]} \epsilon_p = \bar{\epsilon} > 0$. Then, under assumptions $\mathcal{A}$ 4.1- $\mathcal{A}$ 4.4, given ψ_0, the exact solution of the Fokker Planck equation corresponding to $\mathcal{D}_0$, there exists a finite sequence of functions $\{\psi_n\}_{n=1}^{M}$ s.t. $\psi_M = \psi^*$. Moreover, this sequence can be obtained in a recursive fashion starting with ψ_0.

Proof: Let δ_p be such that if $p' \in B_{\delta_p}(p)$ then $||\psi_p - \psi_{p'}|| \leq \frac{\bar{\epsilon}}{2}$. Note that this is possible due to assumption $\mathcal{A}$ 4.4.

Consider the open covering $\bigcup_{p \in [0,1]} B_{\delta_p}(p)$ of the set $[0, 1]$. Since $[0, 1]$ is compact, there exists a finite subcover of $[0, 1]$ given by $\bigcup_{i=1}^{M} B_{\delta_{p_i}}(p_i)$.

Let $\delta_i \equiv \delta_{p_i}(p_i)$ and $\psi_{p_i} \equiv \psi_i$.

Let us assume that ψ_i is known and we need to obtain ψ_{i+1}. By definition, there exists a p^* s.t. $|p^* - p_i| < \delta_i$ and $|p^* - p_{i+1}| < \delta_{i+1}$. Then, it follows from construction that

$$||\psi_i - \psi_{i+1}|| \leq ||\psi_i - \psi_{p^*}|| + ||\psi_{p^*} - \psi_{i+1}|| \leq \bar{\epsilon}. \tag{36}$$

Then, due to Proposition 4.1, starting with ψ_i, it is possible to obtain ψ_{i+1} in a recursive fashion. Note that the above holds for all $i = 0, 1, \cdots, M - 1$. Thus, in this fashion we can obtain the sequence $\{\psi_0, \psi_1, \cdots, \psi_M = \psi^*\}$ recursively starting with ψ_0.

This completes the proof of the proposition.

Q.E.D.

In summary, the development above (the time homotopy and the space homotopy) can be presented as the following algorithm:

- **Step 1:** Find a homotopy of dynamical systems $\mathcal{D}_p$, $p \in [0, 1]$, such that $\mathcal{D}_1$ corresponds to the system of interest and $\mathcal{D}_0$ corresponds to a known system, in the sense that its associated Fokker Planck equation can be solved.

- **Step 2:** Select points $p_i \in [0,1], i = 1, \cdots, M$; that are "sufficiently" close. Let $\mathcal{D}_i$ be the dynamical system corresponding to p_i and ψ_i be the solution to the associated Fokker Planck equation.
- **Step 3:** set i = 1.
- **Step 4:**
  - **(i)** Divide $[0,T]$ into "sufficiently" small intervals $[T_k, T_{k+1}]$, $k = 1, \cdots, N$, $T_1 = T_0, T_N = T$.
  - **(ii)** Set j = 1.
  - **(iii)** Find the solution of of ψ_{i+1} over $[T_j, T_{j+1}]$ using the Galerkin projection equation (22), successive approximations and the converged solution for ψ_{i+1} at T_j and the solution ψ_i over $[T_j, T_{j+1}]$ (using Proposition 4.1).
  - **(iv)** Set j = j+1; **if** $j < N$, go to **Step 4 (iii)**, **otherwise**, go to **Step (5)**.
- **Step 5:** Set $i = i + 1$; **if** $j < M$, go to **Step (4)** , **otherwise**, stop.

The method assumes the knowledge of a homotopy of dynamical systems such that the desired pdf (the solution of the Fokker Planck equation for the dynamical system of interest) can be obtained by smoothly changing a known pdf (the solution of the Fokker Planck equation corresponding to a known dynamical system) by slowly varying the homotopy parameter. This is a design aspect aspect of the problem and is up to the taste of the person solving the equation. For example, consider the duffing oscillator:

$$\ddot{x} = -kx + \epsilon x^3 + w, \tag{37}$$

with $\epsilon = 1$ and w is white noise with unit intensity. In the above problem, the nonlinearity factor ϵ can be considered to be the homotopy parameter. The answer to the problem can be found for the case of $\epsilon = 0$, which is a linear system and the value of ϵ can be increased slowly till it equals unity. However, this is only one of the many ways a homotopy of dynamical systems may be constructed for the same system.

V. Conclusions

In this paper, we have presented a homotopic Galerkin approach to the solution of the Fokker-Planck equation. We have argued that the ideal space to approximate the solution is the $L_2(d\Psi^*)$ space, where $d\Psi^*$ is the probability measure induced on $\Re^n$ by the exact solution to the Fokker-Planck equation, ψ^*. We have shown that if we have a sufficiently close approximation to the exact solution, then we can iteratively converge to the exact solution, starting with the approximate solution, due to a time homotopy in conjunction with the Galerkin projection method. In the case that we do not have a sufficiently close approximation of the exact solution, under the assumption of the existence of a homotopic transformation which allows the exact solution of a known system (in the sense that the Fokker Planck equation associated with it can be solved exactly) to be smoothly changed into the solution of the unknown system, we can recursively obtain the exact solution to the unknown system starting with the exact solution of the known system.

In the introduction we had mentioned that the solution of the Fokker-Planck equation is made difficult due to the three problems of "normality", "positivity" and "domain". In this work, we have addressed the domain aspect of the problem. However, we have to study the normality and positivity aspects of the problem too. We also need to apply the methodology presented in this paper to the solution of various different stochastic dynamical systems in order to understand the "design" aspects of the method.

References

[1] H. Risken, *The Fokker Planck Equation: Methods of Solution and Applications*, Springer Series in Synergetics, vol. 18, Springer-Verlag, Berlin, 1989

[2] N. C. Nigam, *Introduction to Random Vibrations*, Cambridge: MIT Press, 1983

[3] A. M. Ilin and R. Z. Khasminski, "On Equations of Brownian Motion", *Theory Prob. Appl.*, 1964, vol. 9, pp 421-444

[4] H. J. Kushner, "The Cauchy Problem for a Class of Degenerate Parabolic Equations and Asymptotic Properties of Related Diffusion Processes", *J. Diff. Eqns.*, 1969, vol.6, no. 2, pp 209-231.

[5] E. Wong, "The construction of a Class of Markov Processes", *Proc. Symp. Appl. Math.*, 1964, vol. 16, pp 264-276

[6] T. K. Caughey, "Nonlinear Theory of Random Vibrations", *Adv. App. Mech.*, 1971, vol. 9, pp 209-253

[7] Y. K. Wen, "Approximate Method for Nonlinear Random Vibration", *J. Eng. Mech. Div.*, ASCE, 1975, 101(EM4), pp 389-401.

[8] S. H. Crandall, "Perturbation Techniques for Random Vibrations of Nonlinear Systems", *J. Acoust. Soc. Amer.*, 1963, 35(14), pp. 1700-1705

[9] R. C. Booton, "The Analysis of Nonlinear Control Systems with Random Inputs", *Circuit Theory, IRE Trans.*, 1954, vol. 1, pp. 32-34

[10] T. K. Caughey, "Response of Nonlinear String to Random Loading", *J. Appl. Mech.*, 1959, vol. 26, pp. 341-344

[11] R. H. Kraichan, "The Closure Problem of Turbulence Theory", *Symp. Appl. Math.*, 1962, vol. 13, pp. 199-225.

[12] L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis*, Interscience Publishers, New York, 1958

[13] E. Ziedler, *Nonlinear Functional Analysis and its Applications, II/A: Linear Monotone Operators*, Springer-Verlag, Berlin, 1990.

[14] C. Soize, "Steady state solution of Fokker-Planck Equation for Higher Dimensions", *Probab. Eng. Mech*, 1989, vol. 3, pp. 196-206

[15] Q. Liu and H. G. Davies, "The Non-Stationary Response Probability Density Function of Nonlinearly Damped Oscillators Subjected to White Noise Excitations", *J. Sound Vib*, 1990, 139(3), pp. 425-435

[16] G. Muscolino, G. Ricciardi and M. Vasta, "Stationary and Non-Stationary Probability Density Function for Non-linear Oscillators", *Int. J. Nonlinear Mech.*, 1997, 32(6) pp. 1051-64

[17] G-K. Er, "A Consistent Method for the Solution to reduced FPK equation in Statistical Mechanics", *Physica A*, 1999, vol. 262, pp. 118-28

[18] G-K. Er, "Exponential Closure Methods for some Randomly Excited Nonlinear Systems", *Int. J. Nonlinear Mech.*, 2000, 35, pp. 69-78

[19] M. Di Paola and A. Sofi, "Approximate Solution of the Fokker-Planck-Kolmogorov Equation", *Prob. Eng. Mech.*, 2002, vol. 17, pp. 369-384

[20] D. G. Luenberger, "Optimization by Vector Space Methods", *Wiley-Interscience*, New York, 1969

[21] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[22] H. K. Khalil, *Nonlinear Systems, second ed.*, Prentice Hall, NJ, 1996.